Singlet ground state of the bilinear-biquadratic exchange Hamiltonian

A. Tanaka and T. Idogaki
Department of Applied Science, Faculty of Engineering, Kyushu University, Fukuoka 812-81, Japan
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We show that the ground state of the bilinear-biquadratic exchange Hamiltonian is always singlet on a connected finite lattice with an even number of sites, while it is always triplet on one with an odd number of sites, if coefficients of the bilinear term $-J$ and that of the biquadratic term $-J'$ satisfy the condition $J'>J>0$. We also find that the total-spin eigenvalue $S_{\rm tot}$ of the lowest-energy state in each subspace classified by M, an eigenvalue of z component of the total spin, depends on the parity of M, i.e., $S_{\rm tot}=|M|$ for even M and $S_{\rm tot}=|M|+1$ for odd M on an even number site lattice and vice versa on an odd number site lattice. [S0163-1829(97)04941-2]

In some magnetic materials it has been recognized that the higher-order couplings play an important role as well as usual bilinear exchange interactions, and many theoretical and experimental investigations about the effects of these couplings have been performed. For the case of $S=1$, an isotropic model with these couplings is expressed by the following bilinear-biquadratic exchange Hamiltonian:

$$\mathcal{H}_\Lambda=-J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j-J'\sum_{\langle i,j\rangle}(\mathbf{S}_i\cdot\mathbf{S}_j)^2, \tag{1}$$

where summation is taken over all bonds on a lattice Λ. Blume and Hsieh[1] suggested that quadrupole ordering might occur on this system, and many people have discussed the possibility and the properties of this type of ordering.[2–10] In particular, Chen and Levy[6] discussed the possible ordered phases and phase transitions associated with the dipole and quadrupole moments in detail by means of a mean-field approximation and high-temperature series expansion. Their results strongly support the occurrence of quadrupole phase transition for $J'>J>0$.

As far as we know, however, the exact result about the ground state is still rare, except for recent remarkable progress on linear chain systems.[11–17] One of the exact results on general lattice systems is Munro's argument[18] for $J<0$, $J'>0$, which is the straightforward extension of the Lieb-Mattis theorem.[19] Furthermore, this result was extended to the case $J'>J>0$ by Parkinson.[20] His argument is almost exact. However, to determine the total spin eigenvalue, he needed one assumption that the ground state for $J=0$, $J'>0$ is nondegenerate, which was confirmed only by the numerical diagonalization restricted for the finite-size chain systems up to eight sites. Since this region is assumed to be a quadrupole ordering phase at a sufficiently low temperature, it is desirable to discuss the ground state in detail to clarify the nature of this ordering.

In this paper, we show the lattice where Parkinson's assumption is not satisfied and prove that the ground state of the Hamiltonian $\mathcal{H}_\Lambda$ is always singlet in the region $J'>J>0$ on general finite lattice systems with an even number of sites. We also extend the discussion to the lowest-energy state in a subspace classified by an eigenvalue of z component of the total spin and to the case of an odd number of sites.

For a while, we restrict ourselves to a finite bipartite lattice $\Lambda=A+B$, where there is no bond connecting sites $i\in A$ and $j\in A$ or $i\in B$ and $j\in B$. This restriction will be removed later. We denote the number of the site on X by $|X|$ and assume $|A|\geqslant|B|$ without loss of generality. We define the total spin $\mathbf{S}_{\rm tot}=\Sigma_{j\in\Lambda}\mathbf{S}_j$ and denote the eigenvalues of $(\mathbf{S}_{\rm tot})^2$ and $S^z_{\rm tot}$ by $S_{\rm tot}(S_{\rm tot}+1)$ and M, respectively. We define the state $|\chi_m(M)\rangle$, which belongs to the M subspace, in the following manner:

$$|\chi_m(M)\rangle=C\prod_{j\in\Lambda}(S_j^+)^{m_j}|\chi_0\rangle, \tag{2}$$

where $S_j^+=S_j^x+iS_j^y$ is the usual spin raising operator at site j, $|\chi_0\rangle$ is the eigenstate where all S_j^z's have eigenvalue -1, and C is a positive normalization constant. m represents spin configurations $m=\{m_1,m_2,\ldots,m_{|\Lambda|}\}$ with $m_j=0$, 1, 2 providing that $\Sigma_{j\in\Lambda}(m_j-1)=M$.

Munro's argument is as follows. For $J<0$, $J'>0$ the nonzero off-diagonal matrix elements of the Hamiltonian $\mathcal{H}_\Lambda$ are always negative if one uses the basis states $\{U_1|\chi_m(M)\rangle\}$, where $U_1=\exp(-i\pi\Sigma_{j\in A}S_j^z)$. It is easily seen that within M subspace any two states can be connected by some applications of $\mathcal{H}_\Lambda$. Then, from the Perron-Frobenius theorem it is shown that the lowest-energy state in this subspace is unique and positive definite. Following the argument of Lieb and Mattis, it is also shown that this state has $S_{\rm tot}=|M|$ if $|M|\geqslant|A|-|B|$ and $S_{\rm tot}=|A|-|B|$ if $|M|\leqslant|A|-|B|$, therefore the ground states of $\mathcal{H}_\Lambda$ have $S_{\rm tot}=|A|-|B|$ with obvious degeneracy $2S_{\rm tot}+1$.

Parkinson extended Munro's result to $J'>J>0$ by using other basis states $\{U_2|\chi_m(M)\rangle\}$ with $U_2=\exp[-i(\pi/2)\Sigma_{j\in\Lambda}(S_j^z)^2]$. These basis states make all the off-diagonal nonzero matrix elements of $\mathcal{H}_\Lambda$ negative, provided $J'>J>0$. Since $\mathcal{H}_\Lambda$ within M subspace is still irreducible, the lowest-energy state is again unique and positive definite from the Perron-Frobenius theorem. However, one cannot determine $S_{\rm tot}$ of this state immediately because it is positive definite on the basis states $\{U_2|\chi_m(M)\rangle\}$ which are

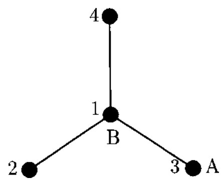

FIG. 1. Four site bipartite lattice Λ_1 with $|A|=3$ and $|B|=1$.

different from those for $J<0$, $J'>0$. To determine $S_{\rm tot}$ of the ground state, Parkinson made one assumption that the ground state is nondegenerate for $J=0$, $J'>0$. As stated in Ref. 20, there appear obvious eigenstates, and also Δ defined by $\Delta=\Sigma_{j\in A}(S_j^z)^2-\Sigma_{j\in B}(S_j^z)^2$ becomes the conserved quantity in this case, which means that $\mathcal{H}_\Lambda$ within M subspace is reducible and one cannot prove the uniqueness of the lowest-energy state within this subspace. If this assumption is satisfied, however, the ground state will automatically be the eigenstate of $(\mathbf{S}_{\rm tot})^2$. In addition, since this ground state will be a positive semidefinite state not only on the basis states for $J<0$, $J'>0$ but on those for $J'>J>0$, it will not be orthogonal to both ground states for each region. Therefore one may be able to conclude that $S_{\rm tot}$ for the ground state for $J'>J>0$ is the same value as that for $J<0$, $J'>0$, i.e., $S_{\rm tot}=|A|-|B|$. There really exists, however, a lattice on which the assumption is not satisfied, as discussed below.

Consider the Hamiltonian $\mathcal{H}_{\Lambda_1}$ with $J=0$, $J'>0$ on the lattice Λ_1 composed of $A=\{2,3,4\}$ and $B=\{1\}$, whose structure is described in Fig. 1. In this case, $M=0$ subspace is found to be decomposed into two disconnected subspaces, one of which corresponds to $\delta=0$, the other to $\delta=2$, where δ is the eigenvalue of Δ. By simple calculation one can find the lowest-energy state

$$|\phi_0\rangle=\sum_m a_m^{(0)}|\chi_m(0)\rangle, \tag{3}$$

$$|\phi_2\rangle=\sum_m a_m^{(2)}|\chi_m(0)\rangle, \tag{4}$$

in the $\delta=0$ and $\delta=2$ subspace, respectively. The coefficients $a_m^{(0)}$ are $3/\sqrt{15}$ for the spin configuration $m=\{m_1,m_2,m_3,m_4\}=\{1,1,1,1\}$ and $-1/\sqrt{15}$ for $m=\{2,0,1,1\}$, $\{2,1,0,1\}$, $\{2,1,1,0\}$, $\{0,2,1,1\}$, $\{0,1,2,1\}$, and $\{0,1,1,2\}$. The coefficients $a_m^{(2)}$ are $1/\sqrt{30}$ for $m=\{1,1,2,0\}$, $\{1,1,0,2\}$, $\{1,2,1,0\}$, $\{1,0,1,2\}$, $\{1,2,0,1\}$, and $\{1,0,2,1\}$ and $-2/\sqrt{30}$ for $m=\{2,2,0,0,\}$, $\{2,0,2,0\}$, $\{2,0,0,2\}$, $\{0,0,2,2\}$, $\{0,2,0,2\}$, and $\{0,2,2,0\}$. These two states have the same energy $-8J'$ and are the ground state. Since Δ and $(\mathbf{S}_{\rm tot})^2$ do not commute each other, these states are not the eigenstates of $(\mathbf{S}_{\rm tot})^2$. The linear combinations of these states become the eigenstates of $(\mathbf{S}_{\rm tot})^2$, and of course we can obtain these states without effort in this simple case, i.e., we have

$$|\tilde{\phi}_0\rangle=\frac{1}{\sqrt{3}}|\phi_0\rangle-\frac{2}{\sqrt{6}}|\phi_2\rangle, \tag{5}$$

$$|\tilde{\phi}_2\rangle=\frac{2}{\sqrt{6}}|\phi_0\rangle+\frac{1}{\sqrt{3}}|\phi_2\rangle, \tag{6}$$

where $|\tilde{\phi}_0\rangle$ and $|\tilde{\phi}_2\rangle$ has $S_{\rm tot}=0$ and 2, respectively. This result means that Parkinson's assumption is not satisfied on the lattice Λ_1.

It is interesting to notice that Eqs. (5) and (6) are rewritten in a positive form

$$|\tilde{\phi}_0\rangle=\frac{1}{\sqrt{3}}\sum_m |a_m^{(0)}|U_2|\chi_m(0)\rangle+\frac{2}{\sqrt{6}}\sum_m |a_m^{(2)}|U_2|\chi_m(0)\rangle, \tag{7}$$

$$|\tilde{\phi}_2\rangle=\frac{2}{\sqrt{6}}\sum_m |a_m^{(0)}|U_1|\chi_m(0)\rangle+\frac{1}{\sqrt{3}}\sum_m |a_m^{(2)}|U_1|\chi_m(0)\rangle. \tag{8}$$

We know that the ground state of $\mathcal{H}_{\Lambda_1}$ with $J'>J>0$ can be expressed as a positive definite state on the basis states $\{U_2|\chi_m(0)\rangle\}$.[20] This implies that the ground state for $J'>J>0$ is not orthogonal to $|\tilde{\phi}_0\rangle$ and has $S_{\rm tot}=0$. On the simple lattice Λ_1, we can obtain the eigenstates of $(\mathbf{S}_{\rm tot})^2$ for $J=0$, $J'>0$ and find that these states are expressed as a positive form. On general lattice systems, however, it is impossible to know what spin states are realized for $J=0$, $J'>0$ because of their possible degeneracy. So we have to go another way to determine $S_{\rm tot}$ for $J'>J>0$.

Hereafter, we consider $\mathcal{H}_\Lambda$ on a connected finite lattice Λ with an even number of sites, which is not required to be bipartite. Since U_2 does not depend on the bipartite structure, the lowest-energy state within M subspace is still unique and positive for $J'>J>0$. Here we consider the lowest-energy state $|\Phi_G(0)\rangle$ only within $M=0$ subspace. Since every energy eigenstate with a given $S_{\rm tot}$ always has a representative in $M\leqslant S_{\rm tot}$ subspace, the global ground state has the same $S_{\rm tot}$ as the lowest-energy state in $M=0$ subspace. Specifically, if this lowest-energy state has $S_{\rm tot}=0$, then this is the unique ground state. If within $M=0$ subspace there exists a state $|\tilde{\Phi}(0)\rangle$ which is an eigenstate of $(\mathbf{S}_{\rm tot})^2$ with eigenvalue $\tilde{S}_{\rm tot}(\tilde{S}_{\rm tot}+1)$ and is a positive semidefinite state on the basis states $\{U_2|\chi_m(0)\rangle\}$, then we are able to conclude $S_{\rm tot}=\tilde{S}_{\rm tot}$ by using the nonorthogonality between $|\Phi_G(0)\rangle$ and $|\tilde{\Phi}(0)\rangle$. In fact, we can find a positive semidefinite state $|\tilde{\Phi}(0)\rangle$ with $\tilde{S}_{\rm tot}=0$, as discussed below. Therefore it can be proved that the ground state for this region is always singlet on general lattice systems with an even number of sites.

From here we prove the existence of a positive semidefinite state $|\tilde{\Phi}(0)\rangle$ on the basis states $\{U_2|\chi_m(0)\rangle\}$ with $\tilde{S}_{\rm tot}=0$. We denote by Ω a set whose elements are pairs of sites:

$$\Omega=\{(j_1,j_2),(j_3,j_4),\ldots,(j_{|\Lambda|-1},j_{|\Lambda|})\},$$

where $j_k\in\Lambda$ and $j_k\neq j_l$ for $k\neq l$. We denote by $|(j_k,j_l)\rangle_0$ the singlet state on the site j_k,j_l:

$$|(j_k,j_l)\rangle_0=|+1\rangle_{j_k}\otimes|-1\rangle_{j_l}-|0\rangle_{j_k}\otimes|0\rangle_{j_l}+|-1\rangle_{j_k}\otimes|+1\rangle_{j_l}, \tag{9}$$

where $S_{j_k}^z|\sigma\rangle_{j_k}=\sigma|\sigma\rangle_{j_k}$. Now, using these singlet states, we define $|\tilde{\Phi}(0)\rangle$ as follows:

$$|\tilde{\Phi}(0)\rangle=(-1)^{|\Lambda|/2}\bigotimes_{(j_k,j_l)\in\Omega}|(j_k,j_l)\rangle_0. \tag{10}$$

Since $|(j_k,j_l)\rangle_0$ is the singlet state, the relation

$$(S^+_{j_l}+S^+_{j_k})|(j_k,j_l)\rangle_0=0, \tag{11}$$

is always satisfied and therefore

$$S^+_{\rm tot}|\tilde{\Phi}(0)\rangle=\sum_{(j_k,j_l)\in\Omega}(S^+_{j_l}+S^+_{j_k})|\tilde{\Phi}(0)\rangle=0. \tag{12}$$

By noting $(\mathbf{S}_{\rm tot})^2=(S^z_{\rm tot})^2+S^z_{\rm tot}+S^-_{\rm tot}S^+_{\rm tot}$, we obtain

$$(\mathbf{S}_{\rm tot})^2|\tilde{\Phi}(0)\rangle=0. \tag{13}$$

On the other hand, if we use $U_2(j_k,j_l)$ defined by

$$U_2(j_k,j_l)=e^{-i(\pi/2)[(S^z_{j_k})^2+(S^z_{j_l})^2]}, \tag{14}$$

Eq. (9) is rewritten as

$$|(j_k,j_l)\rangle_0=-U_2(j_k,j_l)(|+1\rangle_{j_k}\otimes|-1\rangle_{j_l}+|0\rangle_{j_k}\otimes|0\rangle_{j_l}+|-1\rangle_{j_k}\otimes|+1\rangle_{j_l}). \tag{15}$$

By substituting Eq. (15) into Eq. (10) we can express $|\tilde{\Phi}(0)\rangle$ as

$$|\tilde{\Phi}(0)\rangle=U_2\sum_m b_m|\chi_m(0)\rangle,\quad b_m\geqslant 0, \tag{16}$$

with the definition of $|\chi_m(M)\rangle$ given by Eq. (2). From Eqs. (13) and (16), we found that $|\tilde{\Phi}(0)\rangle$ has $\tilde{S}_{\rm tot}=0$ and is positive semidefinite on the basis states $\{U_2|\chi_m(0)\rangle\}$. Now, the proof of the singlet ground state on a lattice with an even number of sites is complete.

We extend the discussion above to the lowest energy state $|\Phi_G(M)\rangle$ within $M>0$ subspace. We define the states $|(j_k,j_l)\rangle_1$ and $|(j_k,j_l)\rangle_2$ as

$$|(j_k,j_l)\rangle_1=|+1\rangle_{j_k}\otimes|0\rangle_{j_l}+|0\rangle_{j_k}\otimes|+1\rangle_{j_l}, \tag{17}$$

$$|(j_k,j_l)\rangle_2=|+1\rangle_{j_k}\otimes|+1\rangle_{j_l}. \tag{18}$$

It is noted that these states can be rewritten as

$$|(j_k,j_l)\rangle_1=iU_2(j_k,j_l)(|+1\rangle_{j_k}\otimes|0\rangle_{j_l}+|0\rangle_{j_k}\otimes|+1\rangle_{j_l}), \tag{19}$$

$$|(j_k,j_l)\rangle_2=-U_2(j_k,j_l)|+1\rangle_{j_k}\otimes|+1\rangle_{j_l}. \tag{20}$$

In the case of even M, we decompose Ω into two disjoint subsets $\Omega_1=\{(j_1,j_2),\ldots,(j_{M-1},j_M)\}$ and $\Omega_2=\{(j_{M+1},j_{M+2}),\ldots,(j_{|\Lambda|-1},j_{|\Lambda|})\}$ and define the state $|\tilde{\Phi}(M)\rangle$ as

$$|\tilde{\Phi}(M)\rangle=(-1)^{|\Lambda|/2}\bigotimes_{(j_k,j_l)\in\Omega_1}|(j_k,j_l)\rangle_2\bigotimes_{(j'_k,j'_l)\in\Omega_2}|(j'_k,j'_l)\rangle_0. \tag{21}$$

We have

$$(\mathbf{S}_{\rm tot})^2|\tilde{\Phi}(M)\rangle=M(M+1)|\tilde{\Phi}(M)\rangle, \tag{22}$$

by noting that

$$(S^+_{j_l}+S^+_{j_k})|(j_k,j_l)\rangle_2=0, \tag{23}$$

$$S^z_{\rm tot}|\tilde{\Phi}(M)\rangle=M|\tilde{\Phi}(M)\rangle. \tag{24}$$

This implies $\tilde{S}_{\rm tot}=M$. Furthermore, by substituting Eqs. (15) and (20) into Eq. (21) we can express $|\tilde{\Phi}(M)\rangle$ as a similar form to Eq. (16). Therefore, the lowest-energy state in even M subspace has $S_{\rm tot}=\tilde{S}_{\rm tot}=M$.

In the case of odd M, we decompose Ω into two disjoint subsets $\Omega_1=\{(j_1,j_2),\ldots,(j_M,j_{M+1})\}$ and $\Omega_2=\{(j_{M+2},j_{M+3}),\ldots,(j_{|\Lambda|-1},j_{|\Lambda|})\}$ and define the state $|\tilde{\Phi}(M)\rangle$ as

$$|\tilde{\Phi}(M)\rangle=i(-1)^{|\Lambda|/2}\times\sum_{(j_k,j_l)\in\Omega_1}\left[|(j_k,j_l)\rangle_1\bigotimes_{\substack{(j'_k,j'_l)\in\Omega_1\\(j'_k,j'_l)\neq(j_k,j_l)}}|(j'_k,j'_l)\rangle_2\right]\bigotimes_{(j''_k,j''_l)\in\Omega_2}|(j''_k,j''_l)\rangle_0. \tag{25}$$

In this case, by making use of the relations

$$(S^+_{j_l}+S^+_{j_k})|(j_k,j_l)\rangle_1=2\sqrt{2}|(j_k,j_l)\rangle_2, \tag{26}$$

$$(S^-_{j_l}+S^-_{j_k})|(j_k,j_l)\rangle_2=\sqrt{2}|(j_k,j_l)\rangle_1, \tag{27}$$

we obtain

$$S^-_{\rm tot}S^+_{\rm tot}|\tilde{\Phi}(M)\rangle=2(M+1)|\tilde{\Phi}(M)\rangle. \tag{28}$$

Combining the same relation as Eq. (24), we find

$$(\mathbf{S}_{\rm tot})^2|\tilde{\Phi}(M)\rangle=(M+1)(M+2)|\tilde{\Phi}(M)\rangle. \tag{29}$$

This signifies $\tilde{S}_{\rm tot}=M+1$. By substituting Eqs. (15), (19), and (20) into Eq. (25), $|\tilde{\Phi}(M)\rangle$ can be rewritten as the positive semidefinite state again. Therefore, the lowest-energy state in odd M subspace has $S_{\rm tot}=\tilde{S}_{\rm tot}=M+1$.

We briefly comment on the case of a connected finite lattice Λ with an odd number of sites. In this case, we select a site, say j, and make a set Ω, except site j. In $M=0$ subspace, we set the state $|\tilde{\Phi}(0)\rangle$ as

$$|\tilde{\Phi}(0)\rangle=(-1)^{(|\Lambda|-1)/2}|0\rangle_j\bigotimes_{(j_k,j_l)\in\Omega}|(j_k,j_l)\rangle_0. \tag{30}$$

It can be easily seen that this state has $\tilde{S}_{\rm tot}=1$ and is positive semidefinite. Therefore, we can prove that the ground state on a lattice with an odd number of sites is triplet.

In the case of odd $M>0$ subspace, we decompose Ω into $\Omega_1=\{(j_1,j_2),\ldots,(j_{M-2},j_{M-1})\}$ and $\Omega_2=\{(j_M,j_{M+1}),\ldots,(j_{|\Lambda|-2},j_{|\Lambda|-1})\}$ and set the state $|\tilde{\Phi}(M)\rangle$ as

$$|\tilde{\Phi}(M)\rangle=-i(-1)^{(|\Lambda|-1)/2}|1\rangle_j\bigotimes_{(j_k,j_l)\in\Omega_1}|(j_k,j_l)\rangle_2\bigotimes_{(j'_k,j'_l)\in\Omega_2}|(j'_k,j'_l)\rangle_0. \tag{31}$$

In the case of even $M>0$ subspace, we decompose Ω into $\Omega_1=\{(j_1,j_2),\ldots,(j_{M-1},j_M)\}$ and $\Omega_2=\{(j_{M+1},j_{M+2}),\ldots,(j_{|\Lambda|-2},j_{|\Lambda|-1})\}$ and set the state $|\widetilde{\Phi}(M)\rangle$ as

$$|\widetilde{\Phi}(M)\rangle=(-1)^{(|\Lambda|-1)/2}|0\rangle_j \bigotimes_{(j_k,j_l)\in\Omega_1}|(j_k,j_l)\rangle_2 \bigotimes_{(j'_k,j'_l)\in\Omega_2}|(j'_k,j'_l)\rangle_0+(-1)^{(|\Lambda|-1)/2}|1\rangle_j \times\sum_{(j_k,j_l)\in\Omega_1}\left[|(j_k,j_l)\rangle_1 \bigotimes_{\substack{(j'_k,j'_l)\in\Omega_1\\(j'_k,j'_l)\neq(j_k,j_l)}}|(j'_k,j'_l)\rangle_2\right] \bigotimes_{(j''_k,j''_l)\in\Omega_2}|(j''_k,j''_l)\rangle_0 . \tag{32}$$

Following the same way in the case of an even number of sites, we can find that these states have $\widetilde{S}_{\text{tot}}=M$ or $\widetilde{S}_{\text{tot}}=M+1$ in odd or even M subspace, respectively, and are rewritten in positive semidefinite form. Therefore, we reach the conclusion that on a lattice with an odd number of sites the lowest-energy state in odd or even M subspace has $S_{\text{tot}}=M$ or $S_{\text{tot}}=M+1$, respectively.

In Fig. 2 we summarize the known results. The ferromagnetic ground state for $J>0$, $J'<J$ was proved by Aksamit[21] by means of the variational method, though we paid little attention to this region in the present paper. We have proved that in the region $J'>J>0$ the ground state of $\mathcal{H}_\Lambda$ is singlet (triplet) on any connected finite lattice with an even (odd) number of sites. We also obtained the total-spin eigenvalue of the lowest-energy state in $M>0$ subspace. The lowest-energy state in $M<0$ subspace can be treated in the same way. These results are summarized as follows. In the case of a lattice with an even number of sites, the lowest-energy state has $S_{\text{tot}}=|M|$ or $S_{\text{tot}}=|M|+1$ in even or odd M subspace, respectively, and in the case of a lattice with an odd number of sites, that has $S_{\text{tot}}=|M|$ or $S_{\text{tot}}=|M|+1$ in odd or even M subspace, respectively. From our results, it turns out that the first exited state for $J'>J>0$ is *not triplet* (assumed to be quintet) on an even number site lattice.

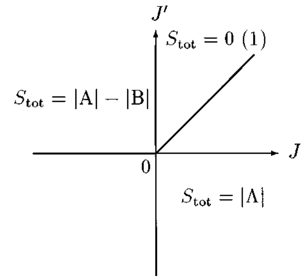

FIG. 2. The ground-state phase diagram of the bilinear-biquadratic exchange Hamiltonian. In the region $J<0$, $J'>0$, the ground state has $S_{\text{tot}}=|A|-|B|$ (the proof is restricted to a bipartite lattice). In the region $J'>J>0$, the ground state is always singlet (triplet) on a lattice with an even (odd) number of sites. In the region $J>0$, $J'<J$, the ground state is ferromagnetic.

We have not discussed the ground state for $J=0$, $J'>0$ on general lattice systems in detail. We assume that in the case of a bipartite lattice with $|A|-|B|=\mathcal{O}(|\Lambda|)$ there is multiple degeneracy on the ground state for this region because $(\mathbf{S}_{\text{tot}})^2$ may connect the two states, one of which has δ, the other of which has $\delta+2$ or $\delta-2$.

Recently, it was shown by Tian[22] that in the case $|A|-|B|=\mathcal{O}(|\Lambda|)$ there are both ferromagnetic and antiferromagnetic long-range orders once the ground states are expressed as the positive definite states on the basis states $\{U_1|\chi_m(M)\rangle\}$. Therefore, on the ground states of the bilinear-biquadratic exchange Hamiltonian with $J<0$, $J'\geqslant 0$ there coexists the ferromagnetic and antiferromagnetic long range orders if $|A|-|B|=\mathcal{O}(|\Lambda|)$. In the present paper we found that the ground state is always singlet on an even number site lattice in the case $J'>J>0$, although the bilinear exchange interaction term is a ferromagnetic one. In this region it is a very interesting problem determining what type of long-range orders occur associated with dipole or quadrupole moments.

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