# Stochastic growth equations and reparametrization invariance

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This article reviews the role of reparametrization invariance (the invariance of the properties of a system with respect to the choice of the co-ordinate system used to describe it) in deriving stochastic equations that describe the growth of surfaces. By imposing reparametrization invariance on a system, the authors identify the physical origin of many of the terms in its growth equations. Both continuum-growth equations for interfaces and equations for the coarse-grained evolution of discrete-lattice models are derived with this method. A detailed analysis of the discrete-lattice case and its small-gradient expansion provides a physical basis for terms found in commonly studied growth equations. The reparametrization-invariant formulation of growth processes also has the advantage of allowing one to model shadowing effects that are lost in the no-overhang approximation and to conserve underlying symmetries of the system that are lost in a small-gradient expansion. Finally, a knowledge of the full equation of motion, beyond the lowest-order gradient expansion, may be relevant in problems where the usual perturbative renormalization methods fail. [S0034-6861(96)00104-3]

## CONTENTS



## I. INTRODUCTION

There has been a great deal of recent work on the formation, growth, and geometry of interfaces (Family and Vicsek, 1991; Meakin, 1993; Barabási and Stanley, 1995; Halpin-Healy and Zhang, 1995). These studies are relevant to a variety of experimental situations including biological growth (Eden, 1958), the propagation of flame fronts (Sivashinsky, 1977, 1979), fluid flow in porous media (Cieplak and Robbins, 1988; Martys *et al.*, 1991), and atomic deposition processes (Messier and Yehoda, 1985; Bales *et al.*, 1990; Tang *et al.*, 1990; Tu and Harris, 1991) such as the technologically important molecular beam epitaxy (MBE). On a more fundamental level, some of these processes are prototypical of far-from-equilibrium physics and do not have a Hamiltonian formulation (Hohenberg and Halperin, 1977). Recent advances have shown that it is nevertheless possible and useful to categorize the systems into universality classes. There have been two principal approaches for the theoretical analysis of such problems. The first (see, e.g., Meakin, 1993) is based on computer simulations of discrete models, and often it provides useful links between analytic theory and experiments. The second approach (Krug and Spohn, 1990; Kardar, 1994; Halpin-Healy and Zhang, 1995) is to describe the dynamical process by stochastic differential equations. This procedure neglects the short length-scale details but provides a coarse-grained de-

scription of the interface that is suitable for characterizing the asymptotic scaling behavior. There are two essential steps to be carried out: first, one must deduce the continuum equation and second, in order to obtain the scaling behavior, the equation needs to be solved or analyzed by renormalization-group techniques (Ma and Mazenko, 1975; Forster *et al.*, 1977; Wolf, 1991; Frey and Täuber, 1994; Sun and Plischke, 1994a, 1994b). The principal theme of this paper is the derivation of the stochastic partial differential equation appropriate to the different physical processes responsible for the growth. We shall show that the principle of reparametrization invariance ($R$ invariance) can be used in a straightforward yet powerful manner to derive the continuum equation. This approach may also be useful in other contexts, such as the study of avalanche dynamics associated with an interface moving in a random medium (Makse, 1995) and interfaces in complex dynamical systems (Kapral *et al.*, 1994).

We restrict ourselves to local-growth processes in which the growth rate is a function of the local properties of the interface. (Nonlocal effects are discussed in the last section in terms of partial differential equations for two coupled fields.) Traditionally, continuum equations have been derived in the no–overhang approximation (Monge representation). The height $h(\vec{r},t)$ of the interface on a reference substrate plane is assumed to be a single-valued function of the lateral coordinates $\vec{r}$ and time $t$. While this assumption is valid and yields the correct scaling behavior in many instances, it has been demonstrated that some essential details are left out in this approach (Keblinski *et al.*, 1994, 1995, 1996). For example, in the columnar-growth regime of sputter deposition of thin films, the merging and regeneration of columns arises due to the presence of outward-flaring columns. This phenomenon, observed even with normally incident depositing atoms, is not captured within the no–overhang approximation. The $R$-invariance principle will allow us to derive the continuum-growth equations without such restrictions—the no–overhang situation can then be obtained as a special case.

Previously, continuum-growth equations have been derived either directly from discrete models by Vvedensky *et al.* (1993) or by using methods based on preserving the symmetries and conservation laws of the system (see, e.g., Hwa and Kardar, 1989; Kardar, 1994). The latter approach has been widely used to identify universality classes of discrete atomistic models (Das Sarma and Tamborenea, 1991; Rácz *et al.*, 1991; Liu and Plischke, 1988; Huse *et al.*, 1990; Das Sarma and Ghaisas, 1992; Kotrla *et al.*, 1992; Amar and Family, 1993; Krug *et al.*, 1993). While this method often unambiguously yields the lowest-order terms in a gradient expansion of the stochastic differential equation, physical considerations have to be invoked to decide whether such terms are indeed present or not.

In the next section we discuss previous work on the derivation of the dynamical equations of surface growth. Section III describes how the $R$-invariance principle can be used to obtain the growth equations in a variety of physical situations. The last section summarizes our results.

The scope of our article is limited—the emphasis is on the theoretical side of surface growth processes and concentrates on the derivation of continuum-growth equations and on the discussion of their geometrical and physical content. We shall mention results which have been obtained elsewhere for these equations and their relationship to physical processes such as MBE or discrete models. Our discussion will certainly fail to be complete, since our primary goal is providing a selfcontained exposition of the derivation of growth equations. We apologize to the many authors whose work is not appropriately mentioned in this article.

A more complete survey of this field can be found in several review articles, each with its own focus. Among the early reviews, the book by Family and Vicsek (1991) and the contributions by Meakin (1988a) and Krug and Spohn (1990) have become standard references. Specifically, the first contains a collection of important reprints with a commentary, the second focuses mostly on the phenomenology of simple models of growth, while the third contains a discussion of analytical approaches. Another early review is the book by Pietronero and Tosatti (1985), which describes fractal growth processes. The effects of a random substrate on wetting phenomena have been considered by Pfeifer *et al.* (1989) and Giugliarelli and Stella (1991, 1994) among others. The review by Halpin-Healy and Zhang (1995) concentrates on the Kardar-Parisi-Zhang (1986) equation and reviews the efforts of the last decade to elucidate its scaling behavior. From this central theme, it extends to cover the field in a pedagogical manner with special emphasis on directed polymers in random media. Numerical simulations of discrete-growth models are nicely reviewed in the work by Meakin (1993). Barabási and Stanley (1995) authored a quite complete elementary book with an extensive list of references. Therein, a lucid discussion of interface growth in random media, including a summary of recent developments, can be found. Finally, a detailed discussion of the phenomenology and some theoretical approaches for the study of MBE growth can be found in the book by Tu and Harris (1991) (see also Tang and Nattermann, 1991; Kessler *et al.*, 1992; Villain *et al.*, 1992; Das Sarma, 1994; Hunt *et al.*, 1994; Pal and Landau, 1994; Siegert and Plischke, 1994; Das Sarma *et al.*, 1996).

## II. PREVIOUS APPROACHES

In this section we shall briefly review the traditional method for deriving local-growth equations. With very few exceptions (Meakin, Ramanlal, *et al.*, 1986; Maritan *et al.*, 1992), this is accomplished within the familiar no–overhang approximation (the Monge representation). Denoting by $h(\vec{x},t)$ the single–valued interface height

function of the lateral coordinates $\vec{x}$ and time $t$, one may generally write a growth equation in the form

$$\frac{\partial h(\vec{x},t)}{\partial t} = G[h(\vec{x},t)] + \eta(\vec{x},t), \tag{1}$$

where $G$ is the deterministic growth term and $\eta$ is the noise. This is essentially Newton's law of motion in which the inertial term has been neglected in comparison with the dissipative force. The neglected term is irrelevant in determining the scaling behavior in the asymptotic regime. This regime, in Fourier space, is related to the behavior of the height-height correlation function in the limit of small frequencies, $\omega \to 0$. While the left-hand side of Eq. (1) is proportional to $\omega$, the inertial term is of order $\omega^2$, and hence it is irrelevant.

Consider a $D$-dimensional substrate of size $L^D$ (the physical case corresponds to a $L \times L$ substrate with $D=2$) and then define the mean height of the growing film and its roughness $W$ by

$$\bar{h}(L,t) = \frac{1}{L^D} \int d^D x h(\vec{x},t), \tag{2a}$$

$$W(L,t) = \left\langle \frac{1}{L^D} \int d^D x [h(\vec{x},t) - \bar{h}(L,t)]^2 \right\rangle^{1/2}, \tag{2b}$$

where $\langle \cdots \rangle$ denotes an average over different realizations of the noise (samples).

Starting from a flat interface (one of the possible initial conditions), it was conjectured by Family and Vicsek (1985) (see also Plischke and Rácz, 1985; Family and Vicsek, 1991) that a scaling of space by a factor $l$ and of time by a factor $l^z$ (dynamic scaling), rescales the roughness $W$ by a factor $l^\alpha$, as appropriate for a self-affine surface (Mandelbrot, 1986), i.e.,

$$W(lL, l^z) = l^\alpha W(L,t), \tag{3}$$

which implies that

$$W(L,t) = L^\alpha f(t/L^z). \tag{4}$$

If, for large $t$ and fixed large $L$ $(t/L^z \to \infty)$, $W$ saturates, then $f(x) \to$ const as $x \to \infty$. However, for fixed large $L$ and $1 \ll t \ll L^z$, one expects that correlations of the height fluctuations are set up only within a distance $t^{1/z}$ and thus $W$ must be independent of $L$. This implies that, for $x \ll 1$,

$$f(x) \sim x^\beta \quad \text{with} \quad \beta = \alpha/z. \tag{5}$$

Thus dynamic scaling postulates that

$$W(L,t) \sim t^\beta \quad 1 \ll t \ll L^z, \tag{6a}$$

$$\sim L^\alpha \quad t \gg L^z. \tag{6b}$$

The roughness exponent $\alpha$ and the dynamic exponent $z$ characterize the self-affine geometry of the surface and its dynamics, respectively.

The above considerations do not include short-time behavior. In that limit one may expect that the height fluctuations are uncorrelated and therefore $h(\vec{x},t) - \bar{h}(t)$ behaves as $\int_0^t \eta(\tau) d\tau$, where $\eta(\tau)$ is a white noise with zero average and $\langle \eta(\tau)\eta(\tau') \rangle \propto \delta(\tau - \tau')$. This leads to $W(L,t) \sim t^{1/2}$ independent of $L$ and is the random-deposition regime. Note that this dynamic scaling is not related to the standard dynamic scaling in critical phenomena. The latter involves correlations which are space-time separated, whereas here they concern the behavior at the same time. Furthermore, after the interfacial roughness has saturated ($t \gg L^z$), one may expect that correlations of fluctuations $\delta h(\vec{x},t) = h(\vec{x},t) - \bar{h}(L,t)$, such as

$$C_L(\vec{x},t;\vec{x}',t') = \langle [\delta h(\vec{x},t) - \delta h(\vec{x}',t')]^2 \rangle, \tag{7}$$

show the more traditional dynamic scaling,

$$C_L(\vec{x},t+\tau;\vec{x}',t) = |\vec{x}-\vec{x}'|^{2\alpha} g\left( \frac{\tau}{|\vec{x}-\vec{x}'|^{z'}}, \frac{|\vec{x}-\vec{x}'|}{L} \right), \tag{8}$$

with $z'$ not necessarily equal to $z$. Note that the exponent $\alpha$ must be the same as before, since

$$\frac{1}{L^{2D}} \int d^D x d^D x' C_L(\vec{x},t;\vec{x}',t) = W^2(L,t) \sim L^{2\alpha} \quad (t \gg L^z). \tag{9}$$

The key idea behind the traditional derivation of $G$ and $\eta$ in Eq. (1) is the identification of the symmetries and conservation laws of the system (Kardar, 1994). The deterministic term $G$ is expanded in powers and combinations of the $h$ field, such that the relevant (in the renormalization-group sense) lowest-order terms that are consistent with the symmetries and conservation laws are retained. The lowest-order contribution is a constant $G_0$, which can always be set to zero by the transformation $h \to h + G_0 t$. This amounts to the use of a reference frame that co-moves with the interface. Symmetries that need to be considered include invariance under translation in time and space, and, along the growth direction, rotational and inversion symmetry about the growth direction. For example, translational invariance in the growth direction, which holds in most practical cases, rules out terms proportional to powers of $h$. An explicit dependence of $G$ on time or position $\vec{x}$ contradicts time or space translational invariance, which usually holds, and can therefore be neglected. In addition it is helpful to classify terms which might occur in the expansion into two categories: those which conserve the number of particles and those which do not. To derive the equation of growth for a conservative dynamics, the key observation (Villain, 1991; Kardar, 1994) is that the deterministic part must have the form of a continuity equation,

$$\frac{\partial h(\vec{x},t)}{\partial t} = -\vec{\nabla} \cdot \vec{j}(\vec{x},t), \tag{10}$$

where the macroscopic current $\vec{j}(\vec{x},t)$ describes the flux of atoms on the surface. The current arises in general from differences in the local chemical potential $\mu(\vec{x},t)$. The relation between $\vec{j}(\vec{x},t)$ and $\mu(\vec{x},t)$,

$$\vec{j}(\vec{x},t) = -\vec{\nabla}\mu(\vec{x},t), \tag{11}$$

describes the fact that atoms drift to regions of minimum chemical potential. The simplest source of chemical potential, gravitational energy with $\mu \propto h$, led Edwards and Wilkinson (EW) (1982) to formulate the equation

$$\frac{\partial h(\vec{x},t)}{\partial t} = \nu\nabla^2 h(\vec{x},t) + \eta(\vec{x},t). \tag{12}$$

The noise here arises due to the stochastic nature of the incoming flux of atoms, and, as such, it is nonconservative. The noise can also result from the probabilistic nature of the diffusion process on the surface or from thermal fluctuations. In the latter cases, the number of atoms on the surface remains the same, and the noise will be called conservative. The noise correlation functions are different in the two cases and are given by

$$\langle \eta(\vec{x},t)\,\eta(\vec{x}',t')\rangle = 2D_0\delta^D(\vec{x}-\vec{x}')\,\delta(t-t'), \quad \text{nonconservative, and} \tag{13}$$

$$\langle \eta(\vec{x},t)\,\eta(\vec{x}',t')\rangle = (-D_1\nabla^2 + D_2\nabla^4)\,\delta^D(\vec{x}-\vec{x}')\,\delta(t-t'), \quad \text{conservative.} \tag{14}$$

We shall not consider the case where the statistics of $\eta$ is not Gaussian (see, e.g., Zhang 1990, and Horváth *et al.*, 1991; Krug, 1991; Lam and Sander, 1993) or where it is long-range correlated in space or in time (see Medina *et al.*, 1989; Peng *et al.*, 1991; Lam *et al.*, 1992).

Usually the effect of gravity on deposition processes is totally negligible with respect to other effects. However, the Edwards–Wilkinson equation also applies to other situations, for example, to the case where atoms are allowed to evaporate from the surface. This effect, which does not conserve the number of particles, is nevertheless described by a term, $\nu\nabla^2 h$, in the growth equation. We repeat here the derivation given by Villain (1991). In general, if the deterministic growth term $G$ describes dynamics that minimizes a potential $V[h]$ [which is a functional of the interface configuration $h(\vec{x},t)$], it will have the form

$$G = -\nu\frac{\delta V(h)}{\delta h(\vec{x},t)}, \tag{15}$$

where $\delta/\delta h(\vec{x},t)$ denotes a functional derivative. The effect of evaporation is to minimize the surface area. Since the excess surface area due to roughness is given, in a small-gradient expansion, by $V[h] = \int d^d x(\vec{\nabla}h)^2$ (Bruinsma and Aeppli, 1984), Eq. (15) directly gives the first term on the right-hand side of Eq. (12). It is interesting to note already how a relevant feature of the process, i.e., whether the number of particles is conserved or not, is lost in the small-gradient expansion of the growth equation. A derivation of the growth equation from the $R$-invariance principle avoids such ambiguities.

Let us return to conservative equations and look for the simplest equation one gets by neglecting gravity. From the above discussion, it is natural to expect that the next term in the expansion of the chemical potential will be a term proportional to the local curvature, $\mu \propto \nabla^2 h$. This term, via Eqs. (10) and (11), will favor again a flux of particles away from the local maxima towards the local minima. Hereafter the resulting relaxation mechanism will be called surface diffusion (Villain, 1991; Siegert and Plischke, 1992, 1994; Das Sarma *et al.*, 1996). Since this dynamics constrains the atoms to stay on the surface, this relaxation will be much slower than the one provided by evaporation. The resulting linear equation is

$$\frac{\partial h(\vec{x},t)}{\partial t} = -\kappa\nabla^4 h(\vec{x},t) + \eta(\vec{x},t), \tag{16}$$

and the same considerations discussed previously apply to the noise term here. The two equations can be combined into one that accounts both for gravity and surface diffusion. In experimental situations of MBE, however, the typical coefficients $\nu$ and $\kappa$ are such that the effects of gravity are relevant only on length scales much bigger than the typical sample size. Note that, while rotational invariance in the substrate plane is satisfied by these terms, the full space-rotational invariance is lost. This is a consequence of the small-gradient expansion in $h$, rather than a physical characteristic of the processes. For example, atoms diffuse on the interface irrespective of the direction we assign to the $\hat{z}$ axis. Again, we will recover the full space-rotational invariance and distinguish terms that satisfy this symmetry and those that do not, once the $R$-invariant form of the equation is derived.

Introducing the lowest-order nonlinear term which one could include in $G$ leads to the Kardar-Parisi-Zhang (KPZ) equation (Kardar *et al.*, 1986),

$$\frac{\partial h(\vec{x},t)}{\partial t} = \nu\nabla^2 h(\vec{x},t) + \frac{\lambda}{2}(\vec{\nabla}h)^2 + \eta(\vec{x},t). \tag{17}$$

Here the mechanism of relaxation of surface fluctuations is the same as that of the EW equation (12). The origin of the nonlinear term lies in the driving force of the deposition process that is perpendicular to the interface (or lateral growth in the discrete ballistic deposition model). (For a review of ballistic deposition, see Meakin (1993). See also Ko and Seno (1994) for an off-lattice simulation.) While the simplest growth term, a constant $G_0$, can be eliminated by choosing a co-moving frame on the interface (see above), the nonlinear term of the KPZ equation cannot. The resulting dynamical process is intrinsically irreversible. Interesting generalizations of the

KPZ equation to a multicomponent model have been studied by Doherty *et al.* (1994).

Among the notable properties of the KPZ equation are its relations to a large number of other problems, including the statistics of directed polymers in random media (Kardar and Zhang, 1987; Fisher and Huse, 1991; Kim *et al.*, 1991; Halpin-Healy and Zhang, 1995) and the Burgers equation for fluid dynamics (Forster *et al.*, 1977). The Galilean invariance in the latter problem has profound consequences on the properties of the KPZ equation and leads to the exponent identity $z+\alpha=2$ (Forster *et al.*, 1977; Meakin, Ramanlal, *et al.*, 1986). This invariance translates the problem of interface growth into an invariance of the equation for an infinitesimal tilt of the substrate plane. This invariance can be seen as a remnant of the full $(D+1)$-dimensional rotational invariance of a growth process occurring in the direction normal to the interface. This feature can be fully appreciated in the *R*-invariant form (Maritan *et al.*, 1992) of the equation, which describes an isotropic growth process driven by a pressure or occurring from the condensation of a vapor. In the gradient expansion leading to Eq. (17), rotational invariance is retained only for infinitesimal transformations. However, as we shall see, there are other mechanisms, which are not rotationally invariant, that lead to a growth equation of the form of Eq. (17) in the small-gradient expansion.

Note that in the KPZ equation the $h\rightarrow -h$ symmetry is broken: there is a definite growth direction. Also, the KPZ equation describes processes in which the number of particles is not conserved. The $(\vec{\nabla}h)^2$ term cannot appear in a situation, such as MBE, where conservation of the number of particles on the interface is expected to hold.

One may invoke dynamic scaling to show that, in general, an equation of the type

$$\frac{\partial h}{\partial t}=(-1)^{(n+1)}K_n(\nabla^2)^n h+\eta$$

with

$$\langle \eta(\vec{x},t)\,\eta(\vec{x}',t')\rangle=2D\Gamma_m\delta^D(\vec{x}-\vec{x}')\,\delta(t-t'),$$

where $m$ takes on the two values 0 or 1 with $\Gamma_0=1$ and $\Gamma_1=-\nabla^2$, has the exponents

$$z=2n \quad \text{and} \quad \alpha=\frac{2(n-m)-D}{2}.$$

Thus, for the Edwards–Wilkinson model (Eq. 12) and for Eq. (16), one obtains ($m=0$ in both cases)

$$z=2,\quad \alpha=\frac{2-D}{2} \quad \text{and} \quad z=4,\quad \alpha=\frac{4-D}{2},$$

respectively. The situation with $\alpha<0$ corresponds to $W(t\rightarrow\infty,L)\sim$const ($\alpha=0$ leads to logarithmic corrections). Since the continuum equation was derived in a power series of gradients of $h$ with only the linear term being retained, the result $\alpha\geqslant 1$ is a matter of concern, since local slopes on a distance of order $l$ scale as $|\nabla h|\sim l^{\alpha-1}$. This suggests that neglected nonlinear terms need to be considered as well [for a discussion on self-similar ($\alpha=1$) and self-affine surfaces, see Mandelbrot, 1986 and Meakin, Coniglio, *et al.* (1986)]. The requirements that the basic symmetries be obeyed and that, in the surface diffusion dominated regime, the continuity equation (10) be satisfied lead to [Lai and Das Sarma (1991)]

$$\frac{\partial h(\vec{x},t)}{\partial t}=\nu\nabla^2 h(\vec{x},t)-\kappa\nabla^4 h(\vec{x},t)+\lambda_1\nabla^2(\vec{\nabla}h)^2 +\lambda_2\vec{\nabla}[\vec{\nabla}h(\vec{\nabla}h)^2]+\eta(\vec{x},t), \tag{18}$$

in which all the terms up to fourth order in the gradient expansion of $h$ have been retained. Physically, $\nu$ is set to zero since it arises from a gravitylike chemical potential, which, as mentioned above and as we will discuss below, is negligible. The requirement that the surface current $\vec{j}$ results from a chemical potential would rule out the $\lambda_2$ term, even though this term is more relevant, from a renormalization-group point of view, than the $\lambda_1$ term. It will be clear from the *R*-invariant form of the equation that $\lambda_1$ arises from the second order term in an expansion of a gravity or a surface-tension term. It is interesting to note that recently (Das Sarma and Kotlyar, 1994; Kim and Das Sarma, 1995) it has been proved that, even if the coefficient of the Laplacian is zero in the starting equation, a $\nu>0$ is generated under renormalization by the $\lambda_2$ term, leading ultimately to the EW behavior for surface fluctuations.

Recently, Vvedensky *et al.* (1993) have derived a master equation for surface dynamics based on a microscopic solid-on-solid model that includes deposition, desorption, and diffusion. In the continuum limit, the growth equation [(18)] is obtained—desorption leads to nonzero $\nu$ and $\lambda$ values while surface diffusion produces the $\kappa$ term.

Similar considerations have been used to derive growth equations for interfaces in a disordered medium (Jiang and Hentschel, 1992; Nattermann *et al.*, 1992; Parisi, 1992; Koplik and Levine, 1985; Jensen and Procaccia, 1991; Sneppen, 1992; Nolle *et al.*, 1993; Olami *et al.*, 1994; Galluccio and Zhang, 1995), such as domain walls in random-field Ising models (Nattermann, 1985; Fisher, 1986; Ji and Robbins, 1991; Narayan and Fisher, 1993) and the invasion of one fluid into another within a porous medium (Nittmann *et al.*, 1985; Stokes *et al.*, 1988; Buldyrev *et al.*, 1992; He *et al.*, 1992; Tang and Leschhorn, 1992, 1993; Leschhorn, 1993; Delker *et al.*, 1996). The disorder serves to pin parts of the interface. This feature is captured by replacing the thermal noise with a quenched random noise generated by the disorder $\eta(\vec{x},h)$ governed by correlations of the form

$$\langle \eta(\vec{x},h)\,\eta(\vec{x}',h')\rangle=2D_0\delta^D(\vec{x}-\vec{x}')\Delta(h-h'), \tag{19}$$

where $\langle\cdots\rangle$ represents an average over different realizations of the randomness and the function $\Delta$ characterizes the nature of the quenched disorder. It is important to note that disorder will break the translational invariance in the $h$ direction. Hence a constant $G_0$ term can no longer be eliminated by choosing the co-moving

frame, since, under the transformation $h(\vec{x},t) \to h(\vec{x},t)+G_0 t$, it will reappear in the argument of the noise $\eta(\vec{x},h) \to \eta(\vec{x},h+G_0 t)$. Here, $G_0$ plays the role of the driving force: for small $G_0$ the interface will eventually find a surface where the values of the pinning forces are strong enough to inhibit further growth. In contrast, if $G_0$ is very large, $h$ in the argument of the noise in the co-moving frame can be neglected with respect to $G_0 t$, and one expects to recover the behavior of the corresponding dynamics without disorder, i.e., $\Delta(h-h') \propto \delta(t-t')$. In practice the interface will move fast enough to sample enough values of the disorder in a small time interval, so that the overall effect is that of a time-dependent noise with correlations of the form of Eq. (13). Henceforth, we shall not deal with surface growth in disordered media (for a good review, see Barabási and Stanley, 1995).

## III. REPARAMETRIZATION INVARIANCE AND EQUATIONS FOR THE GROWTH OF SURFACES

The most general Langevin equation for the evolution of a surface in a $(D+1)$-dimensional space has the form

$$\partial_t \vec{r}(\underline{s},t) = \hat{n}(\underline{s},t)\mathcal{G}[\vec{r}(\underline{s},t)] + \vec{F}(\underline{s},t), \tag{20}$$

where the $(D+1)$-dimensional vector $\vec{r}(\underline{s},t) \equiv \{r_\alpha(\underline{s},t)\}_{\alpha=1}^{D+1}$ runs over the surface as $\underline{s} \equiv \{s^i\}_{i=1}^{D}$, varies in a parameter space. (See Appendix A for a brief summary of the elements of differential geometry.) In Eq. (20) $\hat{n}$ stands for the versor normal to the surface at $\vec{r}$, while $\mathcal{G}$ contains a deterministic growth mechanism that causes growth along the normal $\hat{n}$ to the surface and is a functional of $\vec{r}$ itself. $\vec{F}$ is a random force acting on the surface. Equation (20) derives from Newton's laws in the limit of a massless surface, where the inertial term $\partial_t^2 \vec{r}$ can be ignored with respect to the dissipative force. Note that the time derivative of $\vec{r}$ has to be parallel to the normal to the surface. This is because $t=s^0$ can be regarded as the $D+1$th coordinate, and $(\underline{s},s^0)$ is a curvilinear coordinate system. If the growing surface invades the $(D+1)$-dimensional space, this parametrization is legitimate, since the metric tensor is positive definite. However, $s^0$ is the absolute time, and changes of parametrization cannot involve this variable. This requires that $g_{0,i} \equiv \partial_t \vec{r} \cdot \partial_i \vec{r} = 0 \forall i$, which implies that $\partial_t \vec{r} \perp \partial_i \vec{r}$ for all $i=1,\ldots,D$, and therefore $\partial_t \vec{r} \| \hat{n}$.

Independent of the physical mechanisms entering the various terms of Eq. (20), which specify the form of $\mathcal{G}$ and the properties of $\vec{F}$, this equation has to satisfy the fundamental requirement of reparametrization invariance ($R$ invariance). $R$ invariance requires that only quantities that are independent of the choice of the parametrization $\underline{s}$, such as those referring to the local geometry of the surface, like the curvature, can appear in the equation. As with any other symmetry, reparametrization invariance poses constraints on the possible forms that Eq. (20) can take.

It will be convenient to express the deterministic part of Eq. (20), as well as the noise, as a sum of different terms,

$$\mathcal{G} = \mathcal{G}_a + \mathcal{G}_b + \cdots \quad \text{and} \quad \vec{F} = \vec{F}_a + \vec{F}_b + \cdots .$$

The derivation of the equation can then be split into derivations of the individual terms that are expected to appear in a given physical system.

It is often convenient and sufficient to describe the interface in the Monge form, i.e., with $\vec{r} = (\underline{x}, h(\underline{x}))$. Henceforth, at variance with Sec. II, we will use the notation $\underline{x}$ instead of $\vec{x}$ to stress that the Monge form is one of the many parametrizations one can use. $h(\underline{x})$ is the coordinate in the direction normal to the substrate, and, since it must be a single valued function of $\underline{x}$, the interface cannot have overhangs. In this form Eq. (20) becomes

$$\partial_t h(\underline{x},t) = \sqrt{g}(\mathcal{G}+\eta), \tag{21}$$

where $g = 1+(\underline{\nabla} h)^2$ is the determinant of the metric tensor and the relation $\hat{n} = (-\underline{\nabla} h, 1)/\sqrt{g}$ has been used (see Appendix A). $\mathcal{G}$ and $\eta = \vec{F} \cdot \hat{n}$, as before, are the amplitude of the deterministic force and the noise in the normal direction, respectively [note that $\partial_t h(\underline{x},t)/\sqrt{g}$ is the normal velocity $\hat{n} \cdot \partial_t \vec{r}$—this is easy to prove using Eq. (A14)]. The discussion of the noise term is presented in Sec. III B.

### A. Deterministic evolution

As stressed before, reparametrization invariance requires that $\mathcal{G}$ depends only on intrinsic geometric properties of the interface such as the mean curvature $H$ or, when $\mathcal{G}$ is not rotationally invariant, on scalar products of the normal $\hat{n}$ with some fixed vector $\vec{v}$.

The physical meaning of $\mathcal{G}$ is particularly evident for the case in which it can be derived from a potential:

$$\partial_t \vec{r}(\underline{s},t)\big|_{\text{det}} = -\frac{1}{\sqrt{g(\underline{s})}} \frac{\delta \mathcal{H}[\vec{r}(\underline{s})]}{\delta \vec{r}(\underline{s})}, \tag{22}$$

where $\mathcal{H}$ is an $R$-invariant functional of $\vec{r}$. The $1/\sqrt{g}$ term in Eq. (22) appears since

$$\frac{1}{\sqrt{g(\underline{s})}} \frac{\delta f(\underline{s}')}{\delta f(\underline{s})} = \frac{\delta(\underline{s}-\underline{s}')}{\sqrt{g(\underline{s})}}$$

is the $R$-invariant Dirac's delta function (see Appendix A). In this case, the dynamics tends to minimize the potential energy $\mathcal{H}$ of the surface. Moreover, if the random force is properly chosen (this will be discussed in Sec. III.B.), i.e., if it is not conservative, the system approaches a steady state whose distribution of $\vec{r}$ is given by $\exp\{-\beta \mathcal{H}[\vec{r}]\}$, where $\beta$ is related to the correlations of the noise (Hohenberg and Halperin, 1977).

The $R$-invariance of $\mathcal{H}$ guarantees that the functional derivative in Eq. (22) is a vector parallel to the normal as required by Eq. (20), and the $1/\sqrt{g}$ factor guarantees $R$ invariance of the functional derivative. Indeed $R$ in-

variance of $\mathcal{H}$ implies that $\mathcal{H}[\vec{r}(\underline{s}')]=\mathcal{H}[\vec{r}(\underline{s})]$ for any reparametrization $\underline{s}'(\underline{s})$. For an infinitesimal transformation $\underline{s}'=\underline{s}+\underline{\epsilon}(\underline{s})$,

$$\vec{r}'(\underline{s})\equiv\vec{r}(\underline{s}')=\vec{r}(\underline{s})+\underline{\epsilon}(\underline{s})\cdot\underline{\partial}\vec{r}(\underline{s})+O(\epsilon^2),$$

so that

$$\mathcal{H}[\vec{r}(\underline{s}')]=\mathcal{H}[\vec{r}(\underline{s})]+\int d^D s\,\epsilon^i(\underline{s})\partial_i\vec{r}(\underline{s})\cdot\frac{\delta\mathcal{H}}{\delta\vec{r}(\underline{s})}.$$

Since $\partial_i\vec{r}$ is a vector in the tangent plane of the surface, the second term in this equation vanishes for any function $\underline{\epsilon}$ only if the functional derivative is parallel to the normal $\hat{n}$.

In the Monge form, if $\mathcal{G}$ derives from a potential $\mathcal{H}$, we find (see Appendix B)

$$\mathcal{G}=-\hat{n}\cdot\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\vec{r}}=-\frac{\delta\mathcal{H}}{\delta h}. \tag{23}$$

We now proceed to consider the simplest possible terms that can appear in Eq. (20). We first give the expression in a general parametrization and then discuss the Monge form and the expansion in small gradients of $h(\underline{x},t)$.

### 1. Surface tension

The simplest physically motivated term in the Hamiltonian of a surface is proportional to the total area $\mathcal{A}=\int d^D s\sqrt{g}$ and produces a force that tends to minimize the surface area. This term is usually associated with surface tension. The functional derivative yields (see Appendix C)

$$-\frac{1}{\sqrt{g}}\frac{\delta}{\delta\vec{r}(\underline{s})}\int d^D s\sqrt{g}=\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)\vec{r}(\underline{s})=\underline{\Delta}\vec{r}(\underline{s}), \tag{24}$$

where $\underline{\Delta}$ is the Beltrami–Laplace operator defined in Eq. (A4). For $\mathcal{H}_s=\nu_s\mathcal{A}$, using Eq. (A7), one obtains:

$$\mathcal{G}_s=\nu_s\hat{n}\cdot\underline{\Delta}r=\nu_s H, \tag{25}$$

where $H$ is the mean curvature.

Using Eqs. (23), (25), and $\mathcal{H}_s=\nu_s\int d^D x\sqrt{1+(\underline{\nabla}h)^2}$, one finds, in a small-gradient expansion in the Monge parametrization [Eq. (21)],

$$\sqrt{g}\mathcal{G}_s=\nu_s\sqrt{g}\underline{\nabla}\cdot\frac{\underline{\nabla}h}{\sqrt{g}}=\nu_s\left\{\nabla^2 h(\underline{x},t)+\frac{1}{2}\nabla^2 h(\underline{\nabla}h)^2-\frac{1}{2}\underline{\nabla}\cdot[\underline{\nabla}h(\underline{\nabla}h)^2]+\cdots\right\}. \tag{26}$$

The main physical mechanism that produces a term like this is evaporation. Following Villain (1991) we observe that the evaporation rate will be proportional to the difference between the chemical potential of the solid $\mu_s$ and the vapor $\mu_v$. On the surface the former will depend on the local geometry, and hence

$$\partial_t\vec{r}|_{\text{evap}}=\hat{n}B[\mu_s(\vec{r})-\mu_v].$$

We can then expand $\mu_s$ in powers of the local curvature. Along with the zero-order term, corresponding to the chemical potential for a flat surface, there will appear a term proportional to the curvature $H$ as in the right-hand side of Eq. (25).

### 2. Pressure

The second-simplest geometrical property on which $\mathcal{H}$ may depend is the volume enclosed by the surface, which may be written as $\mathcal{V}=(D+1)^{-1}\int d^D s\sqrt{g(\underline{s})}\vec{r}(\underline{s})\cdot\hat{n}(\underline{s})$. A linear dependence of $\mathcal{H}$ on the volume physically represents a pressure term. If $\mathcal{H}_v=-\lambda\mathcal{V}$, the pressure $\lambda>0$ encourages an increase in the volume, while, if $\lambda<0$, the force in Eq. (22) acts to deflate the volume. The infinitesimal volume variation on the surface element $d\sigma=d^D s\sqrt{g}$ is given by $d\sigma\hat{n}\cdot\delta\vec{r}$, so the functional derivative of $\mathcal{H}_v$ in Eq. (22) gives

$$\mathcal{G}_v=-\hat{n}\cdot\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}_v}{\delta\vec{r}}=\lambda, \tag{27}$$

which in the Monge form becomes

$$\sqrt{g}\mathcal{G}_v=\lambda\sqrt{g}=\lambda[1+\tfrac{1}{2}(\underline{\nabla}h)^2+\cdots]. \tag{28}$$

Equation (20) with $\mathcal{G}=\mathcal{G}_s+\mathcal{G}_v$ is one of the $R$-invariant forms (Maritan *et al.*, 1992) of the KPZ equation,

$$\partial_t\vec{r}(\underline{s},t)|_{\text{det}}=\hat{n}(\nu_s H+\lambda), \tag{29}$$

which in the Monge representation is given by

$$\partial_t h(\underline{x},t)|_{\text{det}}=\nu_s\sqrt{g}\underline{\nabla}\cdot\frac{\underline{\nabla}h}{\sqrt{g}}+\lambda\sqrt{g}. \tag{30}$$

Indeed, to lowest order in the gradient expansion, Eq. (17) is recovered (apart from the constant term $\lambda$, which can be absorbed by redefining $h\to h+\lambda t$). The complete $R$-invariant KPZ equation derives from the Hamiltonian

$$\mathcal{H}_{\text{KPZ}}=\int d^D x(\nu_s\sqrt{g}-\lambda h), \tag{31}$$

which, however, is unbounded as $h\to\infty$. Thus, even with a suitable noise term, Eq. (30) does not have $\exp(-\beta\mathcal{H}_{\text{KPZ}})$ as the equilibrium distribution of $h$. This is not surprising and is related to the presence of a pressure that makes the system grow forever. In a sense, the interface growth is intrinsically irreversible. Note that $\lambda$ couples only to the $k=0$ mode of $h$ in a Fourier expansion. This would suggest that steady-state correlation functions, such as $\langle[h(\underline{x})-h(\underline{y})]^2\rangle$, are independent of $\lambda$. A derivation of the KPZ equation from the functional derivative of a free energy with a volume and a surface term was also obtained by Grossmann *et al.* (1991) in a more complex way. An alternative derivation was also given in Keblinski *et al.* (1996).

### 3. Curvature energy

The potential $\mathcal{H}$ may also depend on the curvature $H$ of the interface. In general, this dependence can be expressed in a power-series expansion (the zeroth-order term has already been considered in Sec. III.A.1.)

$$\mathcal{H}_c=\int d^D s\sqrt{g}(\kappa_1 H+\kappa_2 H^2+\cdots)=\mathcal{H}_{c,1}+\mathcal{H}_{c,2}+\cdots. \tag{32}$$

The physics behind the first term reflects the difference in the mechanical properties of the media divided by the interface. Indeed, for $\kappa_1>0$, large negative curvatures are encouraged while positive ones are depressed. The functional derivative is carried out in Appendix D with the result

$$\mathcal{G}_{c,1}=-\frac{1}{\sqrt{g}}\hat{n}\cdot\frac{\delta\mathcal{H}_{c,1}}{\delta\vec{r}}=\kappa_1\left(H^2-\sum_{i=1}^{D}\lambda_i^2\right), \tag{33}$$

where $\lambda_i$ are the eigenvalues of the matrix of the coefficients of the second fundamental form and express the principal curvatures of the surface. Since $H=\lambda_1$ in $D=1$, $\mathcal{G}_{c,1}$ vanishes. This is a consequence of the Gauss–Bonnet theorem, which states that the integral of the Gaussian curvature $K$ on a closed surface is a constant. Since $H=K$ in $D=1$, the variation of $\mathcal{H}_{c,1}$ is zero.

In terms of $h(\underline{x})$, Eq. (33) takes the form

$$\sqrt{g}\mathcal{G}_{c,1}\simeq\kappa_1\left[(\nabla^2 h)^2-\sum_{i,j=1}^{D}(\partial_i\partial_j h)^2+\cdots\right]. \tag{34}$$

In the small-gradient expansion no linear term in $h$ arises. The curvature term, as expected, breaks the symmetry $h\to -h$ in the growth equation.

Similarly, higher powers of $H$ are easily worked out (see Appendix D). For the $p$th term in Eq. (32) we find

$$\mathcal{G}_{c,p}=-\frac{1}{\sqrt{g}}\hat{n}\cdot\frac{\delta\mathcal{H}_{c,p}}{\delta\vec{r}}$$
$$=\kappa_p\left(H^{p+1}-p\,\underline{\Delta}\,H^{p-1}-pH^{p-1}\sum_{i=1}^{D}\lambda_i^2\right). \tag{35}$$

This equation generalizes Eq. (25), which is the $p=0$ case, and Eq. (33) with $p=1$. The effect of these terms is more transparent in the Monge parametrization. Of particular interest is the $\kappa_2$ term in the expansion of the curvature potential,

$$\sqrt{g}\mathcal{G}_{c,2}=-2\kappa_2(\nabla^2)^2 h+\cdots. \tag{36}$$

### 4. Rotational invariance

All the $\mathcal{G}$'s we have considered so far are invariant under rotations in the $D+1$ space. This implies (unless the noise term breaks this invariance) that the evolution equation in the form of Eq. (21) is invariant under the following (infinitesimal) symmetry transformations,

$$\underline{x}'=\underline{x}-\underline{\epsilon}h(\underline{x},t)$$

and

$$h'(\underline{x}',t)=h(\underline{x},t)+\underline{\epsilon}\cdot\underline{x}.$$

Indeed, it is straightforward to verify that the normal velocities $\partial_t h(\underline{x},t)/\sqrt{g(\underline{x},t)}$ and $\partial_t h'(\underline{x}',t)/\sqrt{g'(\underline{x}'t)}$ are equal.

In the small-gradient expansion of Eq. (30), Eq. (17) is obtained after redefining $h\to h+\lambda t$. As a consequence, the above symmetry transformations become

$$\underline{x}'=\underline{x}-\underline{\epsilon}\lambda t$$

and

$$h'(\underline{x}',t)=h(\underline{x},t)+\underline{\epsilon}\cdot\underline{x}.$$

where $-\underline{\epsilon}h(\underline{x},t)$ has been omitted in the first equation, since it contributes only to the higher-order powers neglected in Eq. (30). These symmetry transformations are exact for the KPZ equation and can also be deduced from the Galilean invariance of the related Burgers' equation (Forster *et al.*, 1977; Halpin-Healy and Zhang, 1995).

### 5. External potentials

In the presence of an external potential, $\mathcal{H}$ depends explicitly on $\vec{r}$. The simplest example is that of a gravitational field. The variation $\delta\mathcal{H}_g$ of the gravitational energy, when $\vec{r}\to\vec{r}+\delta\vec{r}$, is proportional to the variation of the mass $\rho d^D s\sqrt{g}\hat{n}\cdot\delta\vec{r}$ ($\rho$ is the mass density) multiplied by the acceleration $a_g$ of the gravitational field and the "height" $\hat{z}\cdot\vec{r}$, where $\hat{z}$ is the direction of the gravitational field. Then,

$$\delta\mathcal{H}_g=\nu_g\int d^D s\sqrt{g}(\vec{r}\cdot\hat{z})\hat{n}\cdot\delta\vec{r}, \tag{37}$$

where $\nu_g=\rho a_g$. It is easy to see that such a term breaks translational invariance, i.e., the growth equation changes if $\vec{r}(\underline{s})\to\vec{r}(\underline{s})+\vec{r}_0$. This generally applies to any potential $V(\vec{r})$ for which $\mathcal{H}=\int d^D s\sqrt{g}V(\vec{r})$, unless $V(\vec{r})=\nu_s$, which yields a surface-tension term (see Sec. III A.1.), or $V(\vec{r})=-\lambda\hat{n}\cdot\vec{r}/(D+1)$, which is the case of a pressure potential dealt with in Sec. III A.2. Since translational invariance is expected to hold, these terms are not considered. In the Monge parametrization, the gravitational energy, which is proportional to $\int d^D x h^2(\underline{x})$, leads to a linear term in $h$ in the equation of motion.

### 6. Orientational energy

The simplest translational-invariant term that breaks rotational invariance results from considering a potential that depends on the local orientation of the surface:

$$\mathcal{H}_z=-\int d^D s\sqrt{g}\chi(n_z), \tag{38}$$

where $n_z=\hat{n}\cdot\hat{z}$, $\hat{z}$ is some fixed direction in the $(D+1)$-dimensional space and $\chi(x)$ is a generic function. Such a term would result, for example, by imposing a constraint on the slopes of the surface with respect to a reference substrate plane. This is often realized in restricted solid-on-solid (RSOS) models for interface

growth (Kim and Kosterlitz, 1989). These are lattice models in which the height of the surface above the reference plane at two neighboring sites can differ only by a discrete number of units, for example, $\pm 1$. The effect of such a constraint propagates at distances larger than the lattice spacing by discouraging configurations with large slopes (in view of their low "entropic" weight). Hence, at a coarse-grained level, the effect of the constraint can be expressed by a term derived from Eq. (38) in a continuum description. The functional derivative of Eq. (38) is performed in detail in Appendix E with the result

$$\mathcal{G}_z = -\frac{1}{\sqrt{g}}\hat{n}\cdot\frac{\delta\mathcal{H}_z}{\delta\vec{r}} = H\left[n_z\frac{d\chi(n_z)}{dn_z}-\chi(n_z)\right] + \frac{d^2\chi(n_z)}{dn_z{}^2}\partial^i(\hat{z}\cdot\vec{r})\partial_i n_z . \tag{39}$$

The first term in Eq. (39) is the one we would obtain by assuming that $d^D s\sqrt{g}\hat{n}\cdot\hat{z}$, the projection of an infinitesimal area onto the substrate plane, does not change under the transformation $\vec{r}\to\vec{r}+\delta\vec{r}$. In the Monge parametrization [see Eq. (E6)] the growth equation becomes ($n_z = 1/\sqrt{g}$)

$$\begin{aligned}\partial_t h = \sqrt{g}\mathcal{G}_z &= \sqrt{g}\left[\underline{\nabla}\left(\frac{\underline{\nabla}h}{\sqrt{g}}\right)\left(n_z\frac{d\chi(n_z)}{dn_z}-\chi(n_z)\right) + \frac{d^2\chi(n_z)}{dn_z{}^2}\frac{\underline{\nabla}h\cdot\underline{\nabla}}{g}\frac{1}{\sqrt{g}}\right]\\ &= (\chi_1'-\chi_1)[\nabla^2 h+\tfrac{1}{2}\nabla^2 h(\underline{\nabla}h)^2] + \tfrac{1}{2}(\chi_1-\chi_1'-\chi_1'')\underline{\nabla}(\underline{\nabla}h(\underline{\nabla}h)^2)+\cdots,\end{aligned} \tag{40}$$

where the last line contains a small-gradient expansion with $\chi_1\equiv\chi(1), \chi_1'\equiv d\chi(x)/dx|_{x=1}$, etc. Note that, even though the first two terms in Eq. (40) are the same as those in Eq. (26) for the surface tension, the latter equation is rotationally invariant while the former is not, due to the presence of the other terms in Eq. (40). Equation (40) also implies that the linear relaxation of the interface (via the term $\nabla^2 h$) is related to the behavior of $\chi(n_z)$ near $n_z=1$. From our previous discussion, one expects that the effect of the constraint in RSOS models is negligible on flat regions, i.e., $\chi_1\simeq 0$. It is also reasonable to assume that $\chi(n_z)$ is an increasing function of $n_z$ that attains its maximum at $n_z=1$. This would indeed favor flat regions over inclined ones and would lead to a positive coefficient, $\chi_1'-\chi_1>0$, in Eq. (40).

In principle one cannot exclude situations in which $\chi_1'-\chi_1<0$. This would lead to a linear instability in the growth equation of the same form as the one which has been invoked by Mazor *et al.* (1988) to describe the columnar morphology of thin films (see below).

It is interesting to note that in $D=1$ the two nonlinear terms in Eq. (40) have the same form $[=(dh/dx)^2 d^2h/dx^2]$ and the coefficient $\chi_1-\chi_1'-\frac{3}{2}\chi_1''$ may assume both positive and negative values, while $\chi_1'-\chi_1$ remains greater than zero.

Here and in the following, if not stated otherwise, we assume that the direction $\hat{z}$ is the same as that along which $h$ is measured. The case in which a different direction is singled out can be dealt with similarly. A specific example of this will be discussed in the next section.

### 7. The flux of particles and geometric effects

Apart from forces that can be derived from a Hamiltonian, effective forces resulting from purely geometrical effects may also appear in Eq. (20). The simplest such nonconservative term derives from a flux of particles with velocity $\vec{\Phi}$ that reaches the surface and sticks to it. The external flux of incoming particles is the basic source of nonconservative noise in interface growth. For this reason, we consider here only the effect of the average flux $\vec{J}=\langle\vec{\Phi}\rangle$. The fluctuation term $\vec{F}=\vec{\Phi}-\vec{J}$ will be the subject of Sec. III B. The growth rate $\mathcal{G}_J$ produced by $\vec{J}$ is proportional to the flux of $\vec{J}$ through the surface and is exactly a measure of how many particles have been added to the surface:

$$\mathcal{G}_J = -\hat{n}\cdot\vec{J} = J n_J , \tag{41}$$

where $n_J$ is the component of $\hat{n}$ in the $-\vec{J}$ direction, $J=|\vec{J}|$, and the negative sign recalls that $\hat{n}$ and $\vec{J}$ have opposite directions. When growth occurs from a vapor, the average of $\vec{\Phi}$ at the position $\vec{r}$ of the interface is proportional to the normal $\hat{n}$ at $\vec{r}$, i.e., $\langle\vec{\Phi}\rangle=-\lambda\hat{n}$, implying $\mathcal{G}_{\rm vap}=\lambda$. Thus growth in the presence of vapor has the same effect as the pressure term in Sec. 2. On the contrary, when particles arrive at the surface in a collimated beam, this term can be eliminated by a Galilean transformation. Indeed, in the framework of Monge parametrization one gets

$$\partial_t h|_{\rm flux} = \sqrt{g}\mathcal{G}_J = -J_z + \underline{J}_\perp\cdot\vec{\nabla}h, \tag{42}$$

with $\underline{J}_\perp$ being the projection of $\vec{J}$ on the substrate plane. The first term is absorbed into $\partial_t h$ by the transformation $h\to h-J_z t$, while the second disappears once $\underline{x}\to\underline{x}+\underline{J}_\perp t$.

Let us return to the height constraint used in some discrete lattice models of the growth of interfaces. Apart from producing a surface tension term, the constraint also has an effect on the flux term. Let us consider a vertical flux $\vec{J}\|\hat{z}$ for simplicity ($\underline{J}_\perp=0$). On a microscopic scale, deposition can only occur at local minima of the discretized surface. In a coarse-grained picture, minima are rare on steep portions of the interface and deposition would be less probable there. In other words, the constraint should reduce the flux $d\phi = n_J d^D s\sqrt{g}$ through the infinitesimal surface element $d^D s\sqrt{g}$, if $n_J\equiv n_z$ is small. This is easily accounted for by multiplying Eq. (41) by an increasing function of $n_z$, $\Upsilon(n_z)$, which leads to

$$\mathcal{G}_J = J n_z \Upsilon(n_z). \tag{43}$$

In the small-gradient expansion in $h$, this term produces

the nonlinear term of the KPZ equation for any choice of the function $\Upsilon(x)$, provided it is increasing. Indeed, if $\vec{J}\|\hat{z}$, one has

$$\partial_t h|_{\text{flux}} = J\sqrt{g}n_z\Upsilon(n_z) = J\Upsilon[1-(\underline{\nabla}h)^2/2+\cdots),$$

which, in a Taylor expansion, contains the nonlinear term of the KPZ equation with a negative $\lambda$. This result was also found by other means (Krug, 1989; Tang *et al.*, 1992). The effect of a constraint on height differences thus produces both the surface tension, as seen in the previous section, and the nonlinear term of the KPZ equation.

Of course the applicability of Eq. (43) extends to any situation where the probability for an incoming particle to stick on the surface depends on the local inclination of the surface. This can also account for the relation between the ballistic deposition model and the KPZ equation. In this model (Meakin, Ramanlal, *et al.* 1986), particles travel in a straight line and attach at the first site they reach in their trajectories that has a nearest neighbor surface site. The result of this mechanism is a noncompact cluster with a fixed density. Even though overhangs are present in the surface, the description of the process in terms of a single-valued function $h(\underline{x},t)$ is possible, at a mesoscopic level, by considering $h(\underline{x},t)$ as the $z$ coordinate of the highest occupied site for each $\underline{x}$. An incoming particle may either stick at the surface or penetrate into the voids of the structure. In the latter case, the deposition process will not result in an increase of $h(\underline{x},t)$. Particles that arrive on a flat portion of the interface have a higher probability of penetrating into the structure than those arriving on steep ones. This situation would also be modeled by Eq. (43), with a function $\Upsilon(n_z)$, that, in contrast to the previous case, should now be a decreasing function of $n_z$; in the small-gradient expansion, this would finally result in a nonlinear term of the KPZ type with a positive coefficient $\lambda$.

The flux term produces another interesting effect in situations where the atoms cannot be approximated as point particles. Mazor *et al.* (1988) have shown that the finite size of particles plays an important role in MBE experiments in which thin films are grown at intermediate temperatures. The basic observation is that, if the atoms have a radius $\xi$, deposition does not actually occur on the surface but at a distance $\xi$ from it in the normal direction. The growth rate is then proportional to the flux of the beam through a surface $\vec{r}'(\vec{s}) = \vec{r}(\underline{s}) + \xi\hat{n}$ that is displaced by an amount $\xi$ in the normal direction from the actual surface. The growth rate on a surface element $d\sigma = d^D s\sqrt{g}$ is proportional to the flux $d\phi = -\hat{n}'\cdot\vec{J}\sqrt{g'}d^D s$ of $\vec{J}$ through the surface $\vec{r}'$. Here the primes refer to the displaced surface. Since $\hat{n}' = \hat{n}$, the only effect comes through the $\sqrt{g}$ factor. The metric tensor $g'_{ij}$ is obtained by observing that $\partial_i\vec{r}' = \partial_i\vec{r} + \xi\partial_i\hat{n}$, so that $g'_{ij} = \partial_i\vec{r}'\cdot\partial_j\vec{r}' = g_{ij} - 2\xi b_{i,j} + O(\xi^2)$ [see Appendix A, especially Eq. (A5)]. Evaluating the determinant of $g'_{ij}$, we find that Eq. (41) has to be modified to

$$\mathcal{G}_{J'} = \mathcal{G}_J + \mathcal{G}_\xi = \frac{d\phi}{d\sigma} = -\hat{n}\cdot\vec{J}\sqrt{\frac{g'}{g}}$$

$$= -\hat{n}\cdot\vec{J}(1-\xi H) + O(\xi^2). \tag{44}$$

In the equation for $h(\underline{x},t)$ a term,

$$\sqrt{g}\mathcal{G}_\xi = -\nu_\xi\underline{\nabla}\frac{\underline{\nabla}h}{\sqrt{g}} = -\nu_\xi\nabla^2 h + \cdots, \tag{45}$$

must be included in the growth equation, with $\vec{J}\|\hat{z}$ and $\nu_\xi = J\xi$, which is positive. Thus the finite size of the incident particles gives rise to an effective antidiffusive behavior in the growth equation.

An interesting extension of the results of this section arises when considering a beam of particles that is not perpendicular to the substrate. This setting has also been considered by Meakin (1988b) and Krug and Meakin (1989, 1991) for ballistic deposition processes. In this situation one generally expects that a coupling between the interface fluctuations and the flux term arises. Let us consider, for example, how an oblique flux modifies the effect of the nonzero radius of particles just discussed. It is straightforward to derive the small-gradient expansion of Eq. (44) in the case $\vec{J} = (\underline{J}_\perp, J_z)$, where the $z$ direction is normal to the substrate:

$$\sqrt{g}\mathcal{G}_\xi = J_z - \underline{J}_\perp\cdot\vec{\nabla}h - \xi J_z\nabla^2 h + \xi\underline{J}_\perp\cdot\underline{\nabla}h\nabla^2 h + \cdots. \tag{46}$$

As already mentioned, the first two terms are eliminated by an appropriate choice of the reference frame. The third term in Eq. (46) has just been discussed. The last one is new, and it represents the coupling of interface fluctuations with the transverse component of the oblique flux. It has been argued by Marsili *et al.* (1996), applying dynamical renormalization-group techniques (see, e.g., Ma and Mazenko, 1975; Forster *et al.*, 1977; Frey and Täuber, 1994) that this term, in competition with surface diffusion, is responsible for a new scaling behavior of interface fluctuations at intermediate scales. This approach predicts a roughness exponent $\alpha = 1/3$ that is in very good agreement with experiments of MBE, where a value $\alpha = 0.30$–$0.33$ was measured (Herrasti *et al.*, 1992; Salvarezza *et al.*, 1992).

One can imagine other effects arising from an inclined flux. For example, if $\underline{J}_\perp \neq 0$, a new term would also arise from the effect of a constraint on height differences on the flux term described in Eq. (43). Indeed, if $\hat{n}\cdot\vec{J} = n_z J_z + \underline{n}_\perp\cdot\underline{J}_\perp$, one would expect a term $(\underline{J}_\perp\cdot\underline{\nabla}h)(\underline{\nabla}h)^2$ in the gradient expansion. Common wisdom (i.e., power counting), however, suggests that this term will not modify the leading-order scaling behavior of interface fluctuations in the presence of the KPZ nonlinearity $(\underline{\nabla}h)^2$.

### 8. Surface diffusion

In cases in which the binding energy of particles on the surface is large compared to thermal-energy fluctuations, the motion of the particles is constrained to be along the surface. This is actually the case in many ex-

perimental realizations of MBE. A force acting in a direction normal to the surface cannot displace the particles. Its effect is to change the local chemical potential $\mu$. Differences in chemical potential on the surface, in turn, produce a current proportional to the gradient of $\mu$ on the surface (Villain, 1991). The evolution of the surface is governed by the continuity equation relating the particle density $\rho$ and this current $\partial_t \rho \propto \hat{n} \cdot \partial_t \vec{r}$. This yields the left-hand side of Eq. (20), while the divergence of the current gives the right-hand side.

The mathematical translation of this argument is as follows. The constraint that the motion of particles occurs on the surface implies that the volume $\mathcal{V}$ it encloses cannot change. Since

$$\partial_t \mathcal{V} = \int d^D s \frac{\delta \mathcal{V}}{\delta \vec{r}(\underline{s},t)} \partial_t \vec{r}(\underline{s},t) = \int d^D s \sqrt{g} \hat{n} \cdot [\hat{n}\mathcal{G} + \vec{F}(\underline{x},t)], \tag{47}$$

the condition $\partial_t \mathcal{V} = 0$ for the deterministic part of Eq. (20) implies

$$\partial_t \mathcal{V} = 0 \Leftrightarrow \int d^D s \sqrt{g} \mathcal{G} = 0. \tag{48}$$

A sufficient condition for this to hold is

$$\mathcal{G} = -\underline{\underline{\Delta}} \mathcal{F} = -\underline{\mathrm{div}}\, \underline{\mathcal{J}}. \tag{49}$$

In other words volume conservation requires $\mathcal{G}$ to be the (covariant) divergence of a (contravariant) current $\mathcal{J}^i$ [see Eq. A3]. Here $\underline{\mathcal{J}}$ is the surface current of particles, and $\mathcal{F}$ is proportional to the chemical potential. This relaxation mechanism is known as surface diffusion. If this force $\mathcal{F}$ derives from a potential, then

$$\hat{n} \cdot \mathcal{F} = -\frac{1}{\sqrt{g}} \frac{\delta \mathcal{H}}{\delta \vec{r}}. \tag{50}$$

In the previous section we derived forces coming from easily modeled contributions to the potential energy of an interface. It is straightforward to find the appropriate $\mathcal{G}^c$ term in the equation for $\partial_t \vec{r}$ under surface diffusion by applying the Beltrami-Laplace operator to these previously derived terms. In the Monge form the term appearing in Eq. (21) has the form

$$\sqrt{g}\mathcal{G}^c = -\sqrt{g}\underline{\underline{\Delta}}\mathcal{F} = \sqrt{g}\underline{\underline{\Delta}} \frac{\delta \mathcal{H}}{\delta h}, \tag{51}$$

where the second equality holds for growth mechanisms that can be derived from a potential (using the result of Appendix B). When surface diffusion occurs to minimize the surface area, the corresponding term in Eq. (20) reads

$$\mathcal{G}_s^c = -\mu_s \underline{\underline{\Delta}}(\hat{n} \cdot \underline{\underline{\Delta}} \vec{r}) = -\mu_s \underline{\underline{\Delta}} H. \tag{52}$$

This follows from the two previous equations and Eq. (25). This term has been widely used in numerical and analytical studies (Mazor *et al.*, 1988; Golubović and Karunasiri, 1991; Siegert and Plischke, 1992; Sun and Plischke, 1993; Sun and Plischke, 1994a, 1994b). Its expression, in terms of $h(\underline{x},t)$, is

$$\sqrt{g}\mathcal{G}_s^c = -\mu_s \sqrt{g}\underline{\underline{\Delta}}\underline{\nabla} \cdot \frac{\nabla h}{\sqrt{g}} \simeq -\mu_s (\nabla^2)^2 h + \cdots. \tag{53}$$

A pressure term does not produce any surface diffusion, since surface diffusion entails the conservation of the volume. External potentials that depend on $\vec{r}$ can produce surface diffusion. In the simplest case of the gravitational potential (along the $\hat{z}$ direction) we have

$$\mathcal{G}_g^c = \nu_g \hat{z} \cdot \hat{n} H, \tag{54}$$

where $\nu_g = \rho a_g$ is the mass density times the gravitational acceleration. The explicit $\hat{n} \cdot \hat{z}$ factor indicates that this term breaks rotational invariance. $\mathcal{G}_g^c$ has the same form as $\mathcal{G}_\xi$ in Eqs. (44) and (45) but has the opposite sign. In actual experimental situations, however, the gravitational energy is negligible with respect to the binding energy. It can be estimated, for example, that in MBE growth of thin films $\nu_g$ is of the order of $10^{-14}$ of the corresponding coefficient in the $\mathcal{G}_\xi$ term. However, terms of the form

$$\sqrt{g}\mathcal{G}_g^c = \nu_g \sqrt{g}\underline{\underline{\Delta}} h = \nu_g \underline{\nabla} \cdot \frac{\nabla h}{\sqrt{g}} \simeq \nu_g \nabla^2 h + \cdots \tag{55}$$

with a positive $\nu_g$ have often been used in recent publications (see, e.g., Golubović and Karunasiri, 1991). This is principally justified by renormalization-group considerations. Even if this term is not present at a microscopic level, it may be generated in the iteration of the renormalization-group equations from the nonlinear terms.

Surface diffusion can also be induced by a curvature-dependent Hamiltonian or by orientation-dependent potentials in the same manner. The explicit expression is simply given by that for the nonconserved case with the additional Beltrami–Laplace operator.

### B. Stochastic evolution

As already mentioned, the principal source of randomness in interface growth comes from a flux of particles that deposit on the surface. Another kind of noise is produced by thermal fluctuations of the surface. The main difference is that the latter conserves the total volume enclosed by the surface. As discussed previously, the random force $\vec{F} = \hat{n}\eta$ is in the normal direction, and we can take $\langle \eta \rangle = 0$. The properties of $\eta$ will now be discussed for the two different cases of nonconservative and conservative noise. Our main concern is the correlator $\langle \eta(\vec{s},t)\eta(\vec{s}',t') \rangle$. In cases when the statistics of the noise is Gaussian, as is almost always assumed, this correlator specifies the entire distribution of $\eta(\vec{s},t)$. [Non-Gaussian statistics for the noise, especially with long tails, has been shown by Zhang (1990) and Krug (1991) to affect the scaling properties of the interface.]

#### 1. Nonconservative noise

For a flux of particles arriving at the surface with velocity $\vec{\Phi}$, the noise term is given by $\eta = \hat{n} \cdot \vec{F}$, where

$\vec{F}=\vec{\Phi}-\vec{J}$ and $\vec{J}=\langle\vec{\Phi}\rangle$. It describes the fluctuation in the number of particles arriving at the surface. The correlator of $\eta$ is then given by

$$\langle\eta(\vec{s},t)\,\eta(\vec{s}',t')\rangle=n_\alpha(\vec{s},t)n_\beta(\vec{s}',t')\langle F^\alpha(\vec{s},t)F^\beta(\vec{s}',t')\rangle$$
$$=n_\alpha(\vec{s},t)n_\beta(\vec{s}',t')\Gamma^{\alpha\beta}\frac{\delta(\vec{s}-\vec{s}')}{\sqrt{g}}$$
$$\times\delta(t-t'). \tag{56}$$

Here we choose $\vec{F}$ to be delta correlated both in space and time. The more general choice of

$$\langle F^\alpha(\vec{s},t)F^\beta(\vec{s}',t')\rangle=\Gamma^{\alpha\beta}C(\vec{s}-\vec{s}',t-t'),$$

where $C(\underline{s},t)$ is a $R$-invariant function, involves no further complication, so we will use the explicit expression of Eq. (56). The coefficients $\Gamma^{\alpha\beta}$ are symmetric in the indices and specify the geometric properties of the noise. We may distinguish two extreme cases: (i) deposition processes that occur from the condensation of an isotropic vapor, and (ii) deposition that occurs from a collimated beam of particles. The difference between these two possibilities is apparent in the Monge parametrization. In order to discuss this, it is useful to introduce the random field

$$\tilde{\eta}(\underline{x},t)=\sqrt{g}\,\eta(\underline{x},t), \tag{57}$$

which is the quantity that appears explicitly in the equation for $\partial_t h$. The correlation properties of $\tilde{\eta}(\underline{x},t)$ are derived directly from those of $\eta(\underline{x},t)$ discussed above.

From Eq. (56) one easily sees that

$$\langle\tilde{\eta}(\underline{x},t)\,\tilde{\eta}(\underline{x}',t')\rangle=\frac{\Gamma^{zz}-2\partial_i h\Gamma^{iz}+\partial_i h\,\partial_j h\Gamma^{ij}}{\sqrt{g}}$$
$$\times\delta(\underline{x}-\underline{x}')\,\delta(t-t'). \tag{58}$$

If growth occurs from the condensation of an isotropic vapor, we expect that $\vec{F}$ is a random vector with uncorrelated components and $\Gamma^{\alpha\beta}=\Gamma\,\delta^{\alpha\beta}$.[1] In Eq. (58) we find

$$\langle\tilde{\eta}(\underline{x},t)\,\tilde{\eta}(\underline{x}',t')\rangle=\Gamma\sqrt{g}\,\delta(\underline{x}-\underline{x}')\,\delta(t-t'). \tag{59}$$

For growth occurring from a directed flux, one may assume that all components of $\vec{F}$ are independent random variables, so that $\Gamma^{\alpha\beta}=\Gamma_\alpha\delta^{\alpha\beta}$ (no summation on $\alpha$ is assumed here). If rotational invariance is expected for rotations in the substrate plane, we have $\Gamma_i=\Gamma_\parallel$ and

$$\langle\tilde{\eta}(\underline{x},t)\,\tilde{\eta}(\underline{x}',t')\rangle=\frac{\Gamma_z+\Gamma_\parallel(\underline{\nabla}h)^2}{\sqrt{g}}\,\delta(\underline{x}-\underline{x}')\,\delta(t-t'). \tag{60}$$

Note that in-plane correlations are enhanced in regions where $h$ has steep derivatives. For a collimated beam perpendicular to the interface, we have $\Gamma_\parallel\ll\Gamma_z$ and

$$\langle\tilde{\eta}(\underline{x},t)\,\tilde{\eta}(\underline{x}',t')\rangle\cong\frac{\Gamma_z}{\sqrt{g}}\,\delta(\underline{x}-\underline{x}')\,\delta(t-t'), \tag{61}$$

while, if $\Gamma_\parallel\cong\Gamma_z$, we recover Eq. (59). If the randomness only affects the intensity of the beam, $\Gamma^{\alpha\beta}=\Gamma J^\alpha J^\beta$, and Eq. (58) becomes

$$\langle\tilde{\eta}(\underline{x},t)\,\tilde{\eta}(\underline{x}',t')\rangle=\Gamma\frac{(J_z-\underline{\nabla}h\cdot\underline{J}_\perp)^2}{\sqrt{g}}\,\delta(\underline{x}-\underline{x}')$$
$$\times\delta(t-t'), \tag{62}$$

where $\underline{J}_\perp$ is the component of $\vec{J}$ in the substrate plane $\vec{r}=\underline{x}$. In the case of vertical rain, $J_z\gg|\underline{J}_\perp|$, and we recover the previous result of Eq. (61).

The physical meaning of the prefactors of the delta functions is evident if we introduce $\eta_o(\underline{x},t)$ such that $\langle\eta_o(\underline{x},t)\,\eta_o(\underline{x}',t')\rangle=\Gamma\,\delta(\underline{x}-\underline{x}')\,\delta(t-t')$. In condensation from a vapor, [see Eq. (59)], we find that $\tilde{\eta}(\underline{x},t)=g^{1/4}\eta_o(\underline{x},t)$ in Eq. (21). [Since $g$ contains stochastic variables, this is also the correct mathematical interpretation of the correlation in Eq. (59).] The noise is enhanced in regions where $h$ has steep derivatives since the exposed surface area in the substrate element $d^Dx$ is larger by a factor of $\sqrt{g}$. The opposite case is that of growth from a perpendicular beam $\vec{J}=J_z\hat{z}$, in which case $\tilde{\eta}(\underline{x},t)=g^{-1/4}\eta_o(\underline{x},t)$. This is because the flux of $\vec{J}$ through the surface is proportional to $\hat{n}\cdot\hat{z}=1/\sqrt{g}$ and regions with high slopes receive less particles than those that are flatter.

### 2. Conservative noise

Another source of noise comes from thermal fluctuations and from internal degrees of freedom of the interface. In this case the noise is called conservative because it causes no increase of the volume enclosed by the interface. From Eq. (47), this requirement is translated into the condition

$$\partial_t\mathcal{V}|_{\text{noise}}=\int d^Ds\,\sqrt{g}\,\eta=0, \tag{63}$$

where again we have taken $\vec{F}=\hat{n}\,\eta$. This imposes a condition on $\eta$. A general way to let the noise contribution in Eq. (63) vanish is to take

$$\eta=\underline{\text{div}}\,\underline{\zeta},$$

where $\underline{\text{div}}$ is the covariant divergence acting on the vector $\zeta$, which is a delta-correlated noise both in space and time:

$$\langle\zeta_i(\underline{s},t)\zeta^j(\underline{s}',t')\rangle=\Gamma\,\delta_i^j\frac{\delta(\underline{s}-\underline{s}')}{\sqrt{g'}}\,\delta(t-t').$$

Here reparametrization invariance has been satisfied, and the delta function allows the use of $g'=g(s')$ instead of $g(s)$. The correlations of $\eta$ readily follow:

[1]Here we use the Kronecker symbol with both upper indices, which coincides with the metric tensor $g^{\alpha\beta}$ in the $(D+1)$-dimensional space.

$$\langle \eta(\underline{s},t)\,\eta(\underline{s}',t')\rangle = \frac{1}{\sqrt{gg'}}\partial_i\{\sqrt{g}\partial_j'[\sqrt{g'}\langle \zeta^i(\underline{s},t)\zeta^j(\underline{s}',t')\rangle]\}$$

$$= \Gamma\frac{1}{\sqrt{gg'}}\partial_i[\sqrt{g}g^{ij}\partial_j'\,\delta(\underline{s}-\underline{s}')\,\delta(t-t')]$$

$$= -\Gamma\frac{1}{\sqrt{g}}\left[\partial_i\sqrt{g}g^{ij}\partial_j\frac{\delta(\underline{s}-\underline{s}')}{\sqrt{g'}}\,\delta(t-t')\right]$$

$$= -\Gamma\,\underline{\underline{\Delta}}\,\frac{\delta(\underline{s}-\underline{s}')}{\sqrt{g'}}\,\delta(t-t'), \tag{64}$$

where the primed quantities refer to $\underline{s}'$ and the presence of the delta function has been used repeatedly to change from primed to unprimed quantities (note that the operator $\underline{\underline{\Delta}}$ does not act on $g'$). This is the natural generalization in reparametrization-invariant form of the correlator often used in dealing with conserved noise that contains the Laplacian operator acting on a delta function. The expression of the correlator in the Monge parametrization is readily derived from the above expression.

### 3. Approach to equilibrium

In previous sections we have frequently dealt with terms in the deterministic part of the growth equation of the form

$$\frac{\partial}{\partial t}h(\underline{x},t) = -\sqrt{g}\Gamma\frac{\delta\mathcal{H}}{\delta h(\underline{x},t)} + \tilde{\eta}(\underline{x},t), \tag{65}$$

where $\Gamma=1$ for nonconservative dynamics and $\Gamma=-\underline{\underline{\Delta}}$ for conservative dynamics. If the noise is Gaussian with correlation [see Eq. (59)]

$$\langle \tilde{\eta}(\underline{x},t)\,\tilde{\eta}(\underline{x}',t')\rangle = 2T\sqrt{g}\Gamma\,\delta(\underline{x}-\underline{x}')\,\delta(t-t'), \tag{66}$$

one can write the associated Fokker–Planck equation in the form

$$\frac{\partial}{\partial t}P[h(\underline{x}),t] = \int d^D x\,\sqrt{g}\Gamma\frac{\delta}{\delta h(\underline{x},t)}\left[T\frac{\delta}{\delta h(\underline{x},t)} + \frac{\delta\mathcal{H}}{\delta h(\underline{x},t)}\right]P[h(\underline{x}),t]. \tag{67}$$

Equation (67) holds if a suitable regularization has been chosen such that $\delta/\delta h(\underline{x},t)$ commutes with $\sqrt{g}\Gamma$. This is fulfilled, e.g., in the dimensional-regularization scheme commonly used in field-theoretical treatments (see, e.g., Zinn-Justin, 1993).

In Eq. (67) $P[h(\underline{x}),t]$ is the probability functional that yields the probability of the interface configuration $h(\underline{x})$ at time $t$. It is easy to see that the stationary distribution, obtained by setting the right-hand side to zero, is given by

$$P[h(\underline{x}),t\to\infty]\propto\exp\left\{-\frac{\mathcal{H}[h(\underline{x})]}{T}\right\}. \tag{68}$$

The equality of the coefficient in front of the functional derivative and of that in front of the noise correlation is often referred to as the fluctuation-dissipation theorem (see, e.g., Deker and Haake, 1975). When this theorem holds, the stationary distribution is given by Eq. (68). It is interesting to note that the $R$-invariant form of the functional derivative, with the $R$-invariant form of the correlator deriving from an isotropic-growth mechanism ($\Gamma=1$), yields a dynamics that leads to the equilibrium distribution of Eq. (68). This was first noted by Bausch *et al.* (1981). One may apply these considerations to the $R$-invariant form of the KPZ equation describing growth from condensation of a vapor as discussed by Maritan *et al.* (1992). However, as already pointed out, $\mathcal{H}_{\rm KPZ}$ of Eq. (31) is unbounded as $h\to\infty$, and therefore Eq. (68), not being normalizable, is meaningless.

Note also that, if the flux term breaks rotational invariance, as in the case of a collimated vertical beam [see Eq. (62)], the fluctuation-dissipation relation does not hold even if the deterministic part derives from a potential.

Furthermore, we note that, for $\Gamma=-\underline{\underline{\Delta}}$, the deterministic part has the same form discussed in Sec. III A.8 and the correlation of the noise is exactly that derived in Eq. (64). Therefore we can conclude that a deterministic conservative dynamics with a conservative noise leads to the stationary state described by Eq. (68) (provided $P[h,t]$ is normalizable).

## IV. DISCUSSION

We have seen that there are four different mechanisms producing a Laplacian term $\nu\nabla^2 h$ in the small-gradient expansion of Eq. (21). Three of them, the surface tension, an orientation-dependent potential (or a constraint on $n_z$), and surface diffusion induced by gravity, lead to a positive $\nu$ coefficient that drives the evolution towards flatter and flatter surfaces. Note that one can distinguish between these effects only through higher-order terms in the gradient expansion, which may, however, be irrelevant in the renormalization-group sense. The fourth mechanism, related to the finite size of the aggregating particles, gives a negative contribution to the coefficient in front of the Laplacian, so it would produce an instability if acting alone. This is evident since Eq. (45) can formally be derived as a surface diffusion induced by a *negative* gravitational field [$\nu_g=-\nu_\xi$ in Eq. (55)]. This term also strictly conserves the volume enclosed by the surface [while surface tension, Eq. (26), does not] even though it was not derived from conservation considerations.

Secondly, we note that the nonlinear term of the KPZ equation can be derived in one of three ways: from a pressure term in a potential that gives a positive $\lambda$ for a growing surface, from growth due to condensation of a vapor, or from an inclination-dependent factor in the flux term. The latter may result from the effect of a constraint on $n_z$, yielding a negative $\lambda$, in agreement with known results on restricted solid-on-solid models (Meakin, 1993). It has also been argued that such a term

is expected in ballistic deposition but with the opposite effect, i.e., inhibiting growth on flat portions of the surface. A positive $\lambda$ is expected in this case. The change of sign in this coefficient does not change the character of the process dramatically as it does for $\nu$. The value of $\lambda$, however, is a directly measurable quantity (Krug, 1989) since it is related to the inclination dependence of the average velocity of growth. In this way Krug (1989) was able to predict the presence of the nonlinear term of the KPZ equation in various models. Note that this is a criterion based on the global behavior, while our analysis is based on the local properties of the growth process. Another derivation of the KPZ equation for restricted solid-on-solid models is based on its relation with the directed-polymer problem (Kardar and Zhang, 1987; Fisher and Huse, 1991) in a random environment as shown, e.g., by Tang *et al.* (1992).

Equation (36) provides a further physical derivation of a term $(\nabla^2)^2 h$, which has usually been associated with surface diffusion, in the equation for a growing interface. This extends the validity of the results derived in the presence of this term to situations in which the restoring force is derived from a potential corresponding to a surface energy proportional to $H^2$. The expansion of the potential energy of the interface in powers of $H$ will, in general, also contain a constant term $H^0$ proportional to the surface area. Straightforward analysis shows that, in the presence of this term, all higher powers are irrelevant from a renormalization-group point of view. There are situations, as in the dynamics of fluid membranes, where this term is known to be absent. The leading terms would then result from the terms linear and quadratic in $H$. An interesting point for future investigation is an analysis of the effects of the term resulting from a potential linear in $H$, Eq. (34).

Sun *et al.* (1989) have studied an equation with a linear term proportional to $(\nabla^2)^2 h$ and a nonlinear term proportional to $\nabla^2(\underline{\nabla} h)^2$ and conservative noise. The same equation was studied by Lai and Das Sarma (1991) and Kim and Das Sarma (1994) (see also Das Sarma *et al.*, 1996) but with a nonconservative noise, as appropriate to MBE. [Lai and Das Sarma (1991) were the first to introduce the nonlinear $\lambda_2$ term of (18).] We note here that, while the first term could be derived from surface diffusion or from a potential proportional to $H^2$, the nonlinear one is not related to one of the simple mechanisms discussed here. In particular, a derivation of the *conserved* KPZ equation cannot follow the same lines described above. In one case, the KPZ equation comes out as a result of a process that does not conserve the volume. In the second, it derives from the effect of a constraint on height gradients, and one needs to motivate the rather odd choice $\Upsilon(n_z) = -\underline{\Delta}\tilde{\Upsilon}(n_z)$. There are no $R$-invariant potentials that would lead to such a term in the gradient expansion, either for conservative or nonconservative dynamics. It is still possible, however, that such a term is generated dynamically in a renormalization-group procedure.

Note also that, in the noise term, there are prefactors in front of the random field. In a small-gradient expansion these yield a nonlinear term of the KPZ type with a random-valued $\lambda$. It is also important to note that the occurrence of multiplicative noise could significantly change the scaling properties of growing interfaces.

We also discussed the generalization of the fluctuation-dissipation theorem to the $R$-invariant form of the growth equation. We recovered the observation of Bausch *et al.* (1981) that $R$ invariance and rotational invariance are enough to establish fluctuation-dissipation relations for nonconserved dynamics. We also found that this observation generalizes to conserved dynamics (both the noise and the deterministic part being conservative).

Recently, Keblinski, Maritan, Toigo, Koplik, and Banavar (1994) and Keblinski *et al.* (1996) have introduced a simple continuum model that allows for overhangs and an arbitrary topology of the growing interface. The model captures surface diffusion in a natural manner, and, with an appropriate aggregation mechanism, it produces growth normal to the interface. The model equation consists of two parts: the first is conserved order-parameter dynamics that allows for the definition of a topologically unrestricted interface and builds the correct physics of surface diffusion, whereas the second term provides for growth and roughening at the interface.

In the simplest version, their equations are

$$\frac{\partial f(\vec{r},t)}{\partial t} = \Gamma \nabla^2 \frac{\delta F}{\delta f(\vec{r},t)} + I, \tag{69}$$

where

$$F = \int \left[ -\frac{1}{2} f^2 + \frac{f^4}{4} + a(\nabla f)^2 \right] dv, \tag{70}$$

and

$$I = C_1 |\vec{\nabla} f| + D_1 \sqrt{|\vec{\nabla} f|}\, \eta(r,t), \tag{71}$$

where $f$ is an order parameter field, $\eta$ is a Gaussian noise uncorrelated in time and space with a width equal to 1 and mean value 0.

Equation (69) without the $I$ term has a simple interpretation—it is merely the deterministic part of the standard model—B dynamics (Hohenberg and Halperin, 1977) that conserves the order parameter.

The choice of the sign of the coefficient of $f^2$ in the expansion for the free energy [Eq. (70)] corresponds to a temperature lower than $T_c$, so that the two values $\pm 1$ of the order parameter $f(\vec{r},t)$ minimizing the free energy describe the two equilibrium phases of the system.

An interface can naturally be defined as the crossover region between the $f=-1$ and $f=+1$ regions, and, operationally, a point $\vec{r}_i$ is defined to be on the interface when $f(\vec{r}_i,t)=0$.

$a>0$ is the surface-diffusion coefficient and sets the intrinsic length in the system. Indeed the width of the

interface is proportional to $\sqrt{a}$. Also, the effective strength of the surface tension turns out to be proportional to $\sqrt{a}$.

The $I$ term allows for the growth and fluctuation of the interface. The $\vec{\nabla} f$ factor ensures that the growth and fluctuations are operative only in the vicinity of the interface—away from the interface $\vec{\nabla} f$ is effectively zero. The positive coefficients $C_1$ and $D_1$ are the magnitudes of the growth and noise, respectively. The $\vec{\nabla} f$ factor produces growth normal to the interface.

Numerical results (Keblinski, Maritan, Toigo, Koplik, and Banavar, 1994; Keblinski *et al.*, 1994, 1995, 1996) show that this model is in the same universality class as the KPZ equation. The $I$ growth mechanism not only gives growth normal to the interface, but also makes the rate of the growth per unit length constant along the interface. The constant is equal to $C_1 \int_{s_{\min}}^{s_{\max}} |\vec{\nabla} f| ds = 2C_1$, where the integral is performed across the interface and $s_{\min}$ and $s_{\max}$, are defined by $f(s_{\min}) = -1$ and $f(s_{\max}) = 1$ (note that the integral is independent of the shape of the $\vec{\nabla} f$ profile as long as $f$ increases monotonically from $f = -1$ to $f = +1$). This feature leads to KPZ-like behavior.

The model presented above has a local conservation law to avoid the formation of islands with $f = -1$ in a region that has predominantly $f = +1$, and vice versa. However, the conserved and nonconserved model should exhibit interfaces with the same behavior. When the interface is sharp, i.e., $a$ is small, it can be shown (Keblinski *et al.*, 1996) that the non-conserved version of this model is equivalent to Eq. (30) with a noise term of the form of Eq. (57) and a variance given by Eq. (59). Thus this model may be interpreted as a continuum version of the Eden growth model with redistribution of the aggregated particles via surface diffusion. The Eden (1958) model is known to be in the KPZ universality class—the surface diffusion in the limit of large length scales does not change the geometrical properties of the interface but introduces a short-range smoothing mechanism.

The growth of real surfaces is often influenced by nonlocal effects such as screening or shadowing. When the aggregating particles follow linear trajectories, one can expect that, if the roughness is large enough, some parts of the interface are shadowed and therefore do not grow (Tang and Liang, 1993). In order to accommodate this phenomenon, the previous model can be extended (Keblinski *et al.*, 1995, 1996) to incorporate the dynamics of the depositing vapor and nonlocal effects.

The extended model involves two fields $f$ and $\phi$ and is governed by the equations

$$\frac{\partial f(\vec{r},t)}{\partial t} = \nabla^2 \frac{\delta F}{\delta f(\vec{r},t)} + B(\nabla f)^2 \phi(\vec{r},t) + C\sqrt{(\nabla f)^2 \phi}\,\eta(\vec{r},t), \tag{72}$$

$$\frac{\partial \phi(r,t)}{\partial t} = \vec{\nabla}[D\vec{\nabla}\phi(\vec{r}t) - \vec{A}\,\phi(\vec{r},t)] - B(\nabla f)^2 \phi(\vec{r},t), \tag{73}$$

with $F$ again given by Eq. (70). While the first part of Eq. (72) is identical to that described earlier, the growth mechanism is different. Now the growth of the $f$ field occurs at the expense of the $\phi$ field. The $\phi$ field represents the local density of the incoming particles towards the interface, and Eq. (73) describes the dynamics of the depositing vapor. The first part of Eq. (73) is simply the diffusion equation in the presence of an external force $\vec{A}$. In order to analyze the growth arising from ballistic trajectories, $D$ has to be chosen much smaller than $\vec{A}$, so the $\vec{A}\phi$ flux is the primary mechanism for $\phi$ field transport.[2] The aggregation relies on the conversion of the $\phi$ field into the $f$ field as described by the coupling term $B$ in Eqs. (72) and (73). The $\nabla f$ factor in the $B$ term makes the aggregation operative only within an interfacial "skin" region of the aggregate with a width proportional to $\sqrt{a}$. The $\phi$ factor in the aggregation term ensures that the growth occurs only if $\phi > 0$. The B term acts as a sink for the diffusive field $\phi$, and its magnitude is chosen to be sufficiently large to convert all of the $\phi$ into $f$ within the interfacial region, which effectively leads to $\phi \sim 0$ below the interface ($f \sim +1$). Shadowing effects are naturally incorporated in the equations. When the $\phi$ field trajectory intercepts the skin, the $\phi$ field is converted into $f$, and any subsequent interception occurs with $\phi = 0$ and therefore does not lead to the growth of the shadowed part of the interface. Note that nonlocal effects are incorporated in a local way in Eqs. (72) and (73). One does not need to monitor the geometry of the interface to incorporate shadowing—it is implemented dynamically by the $\phi$ field. The conversion of the $\phi$ field into the $f$ field at the vicinity of the interface does not depend crucially on the particular functional form of the coupling chosen. The rate of growth is effectively equal to the intensity of the incoming $\phi$ field flux. In addition to the growth term, there is a fluctuation term $C$ in Eq. (72). The Gaussian $\eta(\vec{r},t)$ factor is the same as was introduced previously. In this manner the fluctuations in the strength of the incoming $\phi$ flux are incorporated, since the aggregation rate is equal to the intensity of the incoming $\phi$ flux.

Strikingly, the model presented in Eqs. (72) and (73) can be straightforwardly modified (Keblinski *et al.*, 1996) to model diffusion–limited–aggregation type of phenomena (Witten and Sander, 1981; Niemeyer *et al.*. 1984; see also Pietronero and Tosatti, 1985). One just needs to replace the ballistic flux with a diffusive flux that is responsible for the transport of the aggregating field.

The equations are modified to

$$\frac{\partial f(\vec{r},t)}{\partial t} = \nabla^2 \frac{\delta F}{\delta f(\vec{r},t)} + J(\vec{r},t), \tag{74}$$

$$\frac{\partial \phi(r,t)}{\partial t} = D\vec{\nabla}^2 \phi(\vec{r},t) - J(\vec{r},t), \tag{75}$$

[2]Here $D$ is not to be confused with the spatial dimension.

with

$$J(\vec{r},t) = -\vec{\nabla} f \cdot D\vec{\nabla}\phi \cdot \eta(\vec{r},t), \tag{76}$$

where $\eta(\vec{r},t)$ is a Gaussian noise with nonzero average value $V>0$ and width $W$.

The first part of Eq. (74) and the free energy $F$ is the same as described previously. The interaction term $J>0$, leads to the growth of $f$ and decay of $\phi$ such that $f+\phi$ is a conserved quantity and changes only due to the sources of the $\phi$ field at the boundary. The $R$-invariant form of the growth equation provides an interesting alternative approach to avoid the no-overhang approximation, even though it cannot describe nonlocal effects like shadowing in ballistic aggregation and screening in the diffusion-limited-aggregation models.

The inadequacy of the no–overhang approximation is just one of the reasons why one may want to rely on the $R$-invariant formulation of growth problems. For example, the $R$-invariant form of the equation displays the full invariance properties with respect to space translations and rotations and the conservation laws which the process satisfies. These are lost, for any but infinitesimal transformations, once one restricts attention to the lowest-order terms in the gradient expansion. These invariances play a crucial role in the implementation of renormalization-group approaches around the lower critical dimension [see, e.g., Bausch *et al.* (1981)] because they provide conditions for the renormalizability of the theory.

A second situation in which a full $R$-invariant form of the growth equation would be preferable to the lowest-order gradient expansion arises when the scaling behavior is determined by a strong-coupling fixed point. In order to appreciate this situation it is preferable to sketch briefly the standard approach of the perturbative dynamical renormalization group (Ma and Mazenko, 1975). Let us consider the Langevin equation

$$\partial_t h(\underline{x},t) = \hat{\mathcal{L}} h(\underline{x},t) + \lambda \hat{\mathcal{N}}[h(\underline{x},t)] + \eta(\underline{x},t),$$

where $\hat{\mathcal{L}}$ is a linear differential operator in the $\underline{x}$ variable and $\hat{\mathcal{N}}$ is a nonlinear combination of $h$ and gradients. As an example, in the KPZ equation one has $\hat{\mathcal{L}}=\nabla^2$ and $\hat{\mathcal{N}}[h]=(\underline{\nabla} h)^2$. The noise is Gaussian with zero mean, and $\langle \eta(\underline{x},t)\,\eta(\underline{x}',t)\rangle = 2\Gamma\,\delta^D(\underline{x}-\underline{x}')\,\delta(t-t')$. If $\lambda=0$, the equation can easily be solved in Fourier space. It is therefore easy to find the exponents $\alpha_0$ and $z_0$ that characterize the scaling in the linear theory: under a rescaling of length by a factor of $l$, the time will scale by a factor $l^{z_0}$, and $h$ will acquire a factor $l^{\alpha_0}$. One can then analyze the effect of a small nonlinearity ($\lambda \ll 1$) on the dynamics. If the above discussed change of scale in the linear theory affects the nonlinearity with a factor $l^y$ (i.e., if $\hat{\mathcal{N}}[h] \to l^y \hat{\mathcal{N}}[h]$ as $\underline{x} \to l\underline{x}$, $t \to l^{z_0} t$, and $h \to l^{\alpha_0} h$) and $y<0$, one can conclude that the nonlinearity is irrelevant. On the other hand, if $y>0$, one concludes that the effect of the nonlinearity will increase as the scale increases and will eventually dominate the large-scale behavior of interface fluctuations. It usually happens that $y$ is a decreasing function of $D$, the dimensionality of the substrate and there is a dimension $D_c$ below which the nonlinearity is relevant. In this case, the full program of the dynamic renormalization group becomes necessary. Since the method, beding based on a perturbation expansion, can in principle treat only small nonlinearities, it usually provides estimates of the exponents close to the dimension $D_c$, where one can assume that, since $y$ is small, the fixed point is accessible within the expansion. If this procedure works, i.e., if one finds a stable fixed point whose distance from the fixed point of the linear theory is small when $D_c - D$ is small, one can also conclude that all higher-order terms that have been neglected in the gradient expansion are irrelevant (if $\alpha_0<1$). This whole program, which is a very powerful tool to estimate critical exponents, fails if one finds no stable fixed point or if the nonlinearity turns out to be relevant only if $\lambda$ is bigger than a critical value $\lambda_c$. The nature of the phase for $\lambda>\lambda_c$ turns out to be outside the range of perturbative methods, and in this situation one has no reason to neglect the higher-order terms in the gradient expansion.

This situation is realized in the most-studied model of growth, namely the KPZ equation, for which, above the substrate dimension $D_c=2$ (which is also the physical dimension in which one is interested), the nonlinearity becomes relevant only if $\lambda>\lambda_c$.

There is a general consensus that different models for nonequilibrium dynamics, such as restricted solid-on-solid models (Meakin, 1993), the Eden model (Eden, 1958), and directed polymers in random media (Halpin-Healy and Zhang, 1995), fall in the same universality class. Such an expectation is mainly based on the observation that the small-gradient expansion of these models contains the terms in the KPZ equation. While analytical and numerical results in $D=1$ almost unambiguously support this expectation, there is no reason, in principle, to believe that in $D \geqslant 2$ the strong coupling regime of these models is described by the same exponents. In fact, as we have shown, the continuum equations for the Eden model, for RSOS models, and for ballistic aggregation differ in the higher-order terms of the gradient expansion. A yet different gradient expansion can be obtained for the free energy of directed polymers. Going one step further than the usual KPZ truncation, Marsili and Bray (1996) studied the equation

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \kappa(\vec{\nabla} h)^2 \nabla^2 h + \lambda + \frac{\lambda}{2}(\vec{\nabla} h)^2 + \cdots + \eta. \tag{77}$$

This derives from the small-gradient expansion in the case of RSOS growth, as described in Sec. III.A.6. This equation indeed coincides with Eq. (40) with $\chi_1=0$ and $\chi_1' = -\chi_1'' > 0$. Note that here *both* surface diffusion *and* the growth term have been expanded to the same (second) order. It turns out that a mean-field, infinite-dimensional limit of Eq. (77) can be meaningful only for $\kappa>0$ (Marsili and Bray, 1996). This analysis reveals that steps or bumps of a finite height develop on the surface and that dynamical scaling (Family and Vicsek, 1991) is not satisfied. These results, which differ substantially

from those obtained in the same limit for directed polymers by Derrida and Spohn (1988) (see also Cook and Derrida, 1991), raise doubts on the existence of a single universality class for all these processes. In high dimension the $\kappa$ term turns out to be necessary in order to avoid finite-time singularities that occur in the simpler KPZ equation ($\kappa=0$). These finite-time singularities become even worse in the case of the Eden model or ballistic aggregation. Indeed, the model of Eq. (77) can be generalized to these processes with $\kappa<0$, which describes the suppression of surface tension on steep portions of the interface discussed by Maritan *et al.* (1992). These instabilities suggest that the Monge representation of the interface may not be adequate to describe these processes (Maritan *et al.*, 1992).

Anomalous dynamic scaling has also been reported (Das Sarma *et al.*, 1996) for various models proposed to describe MBE. More precisely, numerical results show that the local-scaling properties, defined in terms of the correlation function, identify an exponent $\alpha_{\rm loc}$, which is different from the one that describes the behavior of the global surface thickness $W(L,t)$ with the size $L$ of the sample at saturation. This anomalous behavior has been related by Schroeder *et al.* (1993), Das Sarma *et al.* (1994) and Krug (1994) to the peculiar statistics of steps (i.e., height differences between neighboring points). The broadness of this distribution which diverges for infinite times and infinite $L$, again raises doubts of the validity of the small-gradient expansion.

In all these situations we believe the concept of reparametrization invariance may prove to be an invaluable starting point for elucidating the correct physics.

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## APPENDIX A: DIFFERENTIAL GEOMETRY

An orthonormal basis is assumed in $(D+1)$-dimensional space, and Greek letters are used for the vector components. Latin letters used as an index refer to the components of vectors in the $D$-dimensional parametrization space. $s^i$ are general curvilinear coordinates that label points on the $D$-dimensional surface. The notation $\partial_i=\partial/\partial s^i$ is used for covariant derivatives.[3] Summation over repeated indices is always assumed. Lastly, for the scalar product in both spaces, a dot is used whereas $\times$ denotes the vector product.

The distance between infinitesimally close points on the surface is given by the first fundamental quadratic form,

$$|d\vec{r}|^2=\partial_i\vec{r}\,ds^i\cdot\partial_j\vec{r}\,ds^j=g_{ij}ds^ids^j. \tag{A1}$$

This defines the metric tensor $g_{ij}=\partial_i\vec{r}\cdot\partial_j\vec{r}$. $g=\det\{g_{ij}\}$ denotes its determinant, while $g^{ij}$ is the inverse,[4] $g_{ik}g^{kj}=\delta^j_i$. The metric tensor and its inverse are also used in the usual way to lower and raise indices, i.e., $v_i=g_{ij}v^j$ or $v^i=g^{ij}v_j$. We also use the notation $\partial^i=g^{ij}\partial_j$.

The only restriction on the choice of the parametrization is that $g\neq 0$, i.e., that $g_{ij}$ is invertible, and this implies also that $\partial_i\vec{r}\neq 0$. The vectors $\partial_i\vec{r}$ lie in the tangent hyperplane, so that the normal versor is given by $\hat{n}=g^{-1/2}\partial_1\vec{r}\times\partial_2\vec{r}\times\cdots\times\partial_D\vec{r}$, where $g^{-1/2}$ ensures normalization $(\vec{v}_1\times\cdots\times\vec{v}_D\equiv\varepsilon_{\alpha_0\alpha_1\ldots\alpha_D}v_{1\alpha_1}\ldots v_{D\alpha_D}$, where $\varepsilon_{\alpha_0\alpha_1\ldots\alpha_D}$ is the completely antisymmetric Levi–Civita tensor and is equal to $(-1)^P$, with $P$ being the order of permutation of $\alpha_0\alpha_1\ldots\alpha_D$ with respect to $1,2,\ldots,D+1)$.

A quantity $T_{i\ldots}{}^{j\cdots}(\underline{s})$ is said to be a tensor if, under the change of parametrization $\underline{s}'(\underline{s})$, it transforms as

$$T'_{i\ldots}{}^{j\cdots}(\underline{s}')=\frac{\partial s^k}{\partial s'^i}\cdots\frac{\partial s'^{j\cdots}}{\partial s^l}T_{k\ldots}{}^{l\cdots}(\underline{s}).$$

From Eq. (A1) one sees that $g_{ij}$ and its inverse $g^{ij}$ are tensors.

A quantity $\varphi(\underline{s})$ is said to be a scalar if $\varphi'(\underline{s}')=\varphi(\underline{s})$. In particular $r_\alpha(\underline{s})$ and $n_\alpha(\underline{s})$, with $\alpha=1,2,\ldots,D+1$, are scalar quantities, while $\partial_i\varphi$ and $\partial^i\varphi$ are particular cases of tensors called covector and vector, respectively.

The invariant surface element is given by $d\sigma=d^Ds\sqrt{g}$, and this implies that the invariant form of the delta function in parameter space is

$$\delta_0(\underline{s}-\underline{s}')=\frac{\delta(\underline{s}-\underline{s}')}{\sqrt{g}}, \tag{A2}$$

where $\delta(\underline{s})$ is the usual delta function in $D$ dimensional space. Thus $\int d\sigma\, f(s)\,\delta_0(\underline{s}-\underline{s}')=f(\underline{s}')$.

For differential calculus, invariant forms of the gradient, divergence, and curl are obtained, which require that the transformation properties of tensors apply. The gradient of a scalar $S$ is simply given by $\partial_iS$, while the divergence of a vector is[5]

$$\underline{\rm div}\ \underline{v}=\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}v^i). \tag{A3}$$

Taking the divergence of the contravariant gradient, one finds the reparametrization-invariant generalization of the Laplacian operator in curved space

[3]We have avoided introducing covariant derivatives in order to maintain as simple as possible an exposition. Thus $\partial_i$ and $\partial^i$ introduced here behave like tensors only when applied to scalar quantities.

[4]The symbol $\delta$ is used here for the Kronecker delta. It will also be used for the Dirac delta function and for functional differentiation.

[5]Indeed, if $\varphi$ and $v^i$ are a scalar and a contravariant field, respectively, Eq. (A3) follows from requiring $\int d^Ds\sqrt{g}v^i\partial_i\varphi=-\int d^Ds\sqrt{g}\varphi\ \underline{\rm div}\ \underline{v}$.

$$\underline{\Delta} = \frac{1}{\sqrt{g}} \partial_i \sqrt{g} \partial^j = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j), \tag{A4}$$

which is known as the Beltrami–Laplace operator.[6]

The curvature $\kappa$ of the surface along a curve $\underline{s}(l)$ is given by $\hat{n} \cdot \partial^2 \vec{r}(\underline{s}(l))/\partial l^2$ ($l$ is the arc length). Since $\hat{n} \cdot \partial_i \vec{r} = 0$, $\kappa$ may be written in terms of the second fundamental quadratic form,

$$\kappa = b_{ij} \frac{ds^i}{dl} \frac{ds^j}{dl}, \tag{A5}$$

where $b_{ij} = \hat{n} \cdot \partial_i \partial_j \vec{r} = -\partial_i \hat{n} \cdot \partial_j \vec{r}$.[7] This defines the principal curvatures (directions) as the eigenvalues $\lambda_i$ (vectors) of $b_i^j$. These are invariant under reparametrization. The mean curvature $H$ is the sum of these and thus equals[8] the trace of $b_i^j$,

$$H = b_i^i = \sum_{i=1}^{D} \lambda_i = -\partial_i \hat{n} \cdot \partial^i \vec{r}. \tag{A6}$$

Another useful definition of $H$ comes from observing that, since $\hat{n} \cdot \partial^i \vec{r} = 0$, $\partial_i (\sqrt{g} \hat{n} \cdot \partial^i \vec{r}) = \sqrt{g} (\partial_i \hat{n}) \cdot \partial^i \vec{r} + \hat{n} \cdot \partial_i (\sqrt{g} \partial^i \vec{r}) = 0$. This implies:

$$H = -\partial_i \hat{n} \cdot \partial^i \vec{r} = \hat{n} \cdot \underline{\Delta} \vec{r}. \tag{A7}$$

The Gaussian curvature is defined as $K = \det\{b_i^j\} = \Pi_i \lambda_i$. Furthermore, it is easy to see that

$$\partial_i \hat{n} = -b_{ij} \partial^j \vec{r}. \tag{A8}$$

Indeed $\partial_i \hat{n} \perp \hat{n}$ and $\partial_i (\hat{n} \cdot \partial_j \vec{r}) = 0$ implies $\partial_i \hat{n} \cdot \partial_j \vec{r} = -\hat{n} \cdot \partial_i \partial_j \vec{r} = -b_{ij}$.

### 1. The Monge form

A particular choice of parametrization is the Monge form,

$$\vec{r} = (\underline{x}, h(\underline{x})), \tag{A9}$$

where $\underline{x}$ is a vector in the $D$-dimensional substrate plane and $h(\underline{x})$ is the height of the surface in the direction $\hat{z}$ perpendicular to this plane. Use of this parametrization implies that no overhangs are present in the surface, since otherwise $h(\underline{x})$ would not be single valued. In this parametrization the metric tensor has the form[9]

$$g_{ij} = \delta_{ij} + \partial_i h \partial_j h \quad \text{and} \quad g^{ij} = \delta_{ij} - \frac{1}{g} \partial_i h \partial_j h, \tag{A10}$$

where

$$g = 1 + (\underline{\nabla} h)^2, \quad \hat{n} = \frac{1}{\sqrt{g}} (-\underline{\nabla} h, 1), \tag{A11}$$

and

$$b^i{}_j = g^{ik} b_{kj} = \partial_i \frac{\partial_j h}{\sqrt{g}}. \tag{A12}$$

Finally, the mean curvature is given by

$$H = b^i{}_i = \underline{\nabla} \frac{\nabla h(\underline{x})}{\sqrt{g}}. \tag{A13}$$

The equation for $h(\underline{x},t)$ is obtained from Eq. (20) by considering the various components of $\vec{r}$. On defining $\vec{r}(\underline{s},t) = (\underline{x}(\underline{s},t), h(\underline{s},t))$, we get

$$\partial_t h(\underline{s},t) = n^z \mathcal{G},$$

$$\partial_t x^i(\underline{s},t) = n^i \mathcal{G},$$

where $n^i$ and $n^z$ are the components of the normal in the directions $\hat{x}^i$ and $\hat{z}$, respectively. These derivatives are evaluated at constant $\underline{s}$, whereas we are interested in the derivative of $h$ at constant $\underline{x}$,

$$\partial_t h(\underline{s},t) = \partial_t h(\underline{x},t) + \frac{\partial h}{\partial x^i} \partial_t x^i(\underline{s},t), \tag{A14}$$

where $h(\underline{x},t) \equiv h(s(\underline{x},t),t)$. From the above equations and $\hat{n} = (-\underline{\nabla} h, 1)/\sqrt{g}$, one readily finds the deterministic part of Eq. (21).

## APPENDIX B: EQUATIONS DERIVED FROM A POTENTIAL IN THE MONGE REPRESENTATION

The property that the functional derivative $\delta \mathcal{H}/\delta \vec{r}$ is orthogonal to the vector $\partial_i \vec{r}$ translates into

$$\frac{\delta \mathcal{H}}{\delta h} \partial_i h + \frac{\delta \mathcal{H}}{\delta x^i} = 0.$$

This allows one to eliminate the functional derivative with respect to $x^i$ in

$$-\sqrt{g} \mathcal{G} = \hat{n} \cdot \frac{\delta \mathcal{H}}{\delta \vec{r}} = n^z \frac{\delta \mathcal{H}}{\delta h} + n^i \frac{\delta \mathcal{H}}{\delta x^i} = \sqrt{g} \frac{\delta \mathcal{H}}{\delta h}$$

and to find Eq. (23).

## APPENDIX C: DERIVATION OF THE GROWTH TERM DUE TO SURFACE ENERGY

In the functional derivative of $g$ with respect to $\vec{r}$ in Eq. (24) we use the fact that $g$ is a determinant and the property

$$\delta \ln \det \hat{M} = \delta \operatorname{tr} \ln \hat{M} = \operatorname{tr} \hat{M}^{-1} \delta \hat{M} \tag{C1}$$

that holds for variations of a matrix $\hat{M}$. This allows one to write

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[6] Equation (A4) can easily be deduced by requiring $\int d\sigma \partial_i \varphi \partial_j \varphi g^{ij} = -\int d\sigma \varphi \underline{\Delta} \varphi$ for any scalar field $\varphi$.

[7] The first form shows that $b$ is a symmetric matrix, whereas the second shows that it is a tensor; hence its eigenvalues are $R$ invariant.

[8] Actually the mean curvature should contain a factor $1/D$ that we disregard for convenience.

[9] In the Monge form, it is convenient to use derivatives, $\partial_i = \partial/\partial x^i$, with respect to the physical coordinates. Even though the upper and lower indices in some of the equations in the Monge form do not match, the summation over repeated (lower) indices is still assumed.

$$\delta\int d^D s\sqrt{g}=\frac{1}{2}\int d^D s\sqrt{g}g^{ij}\delta g_{ji}$$
$$=-\int d^D s\,\partial_i(\sqrt{g}g^{ij}\partial_j\vec{r})\cdot\delta\vec{r}$$
$$=-\int d^D s\sqrt{g}\,\underline{\Delta}\,\vec{r}\cdot\delta\vec{r}$$

for a variation $\delta\vec{r}$ of $\vec{r}$, which readily yields

$$-\frac{1}{\sqrt{g(\underline{s})}}\frac{\delta}{\delta\vec{r}(\underline{s})}\int d^D s'\sqrt{g(\underline{s}')}$$
$$=\frac{1}{\sqrt{g(\underline{s})}}\partial_i(\sqrt{g(\underline{s})}g^{ij}(\underline{s})\partial_j)\vec{r}(\underline{s})=\underline{\Delta}\,\vec{r}(\underline{s}),\qquad\text{(C2)}$$

where Eq. (A4) has been used.

## APPENDIX D: CURVATURE-DEPENDENT POTENTIAL

For a variation $\delta\vec{r}$ in $\vec{r},\hat{n}$ changes to $\hat{n}+\delta\hat{n}$. Since $\delta(\hat{n}\cdot\hat{n})=0$, the variation $\delta\hat{n}$ is normal to $\hat{n}$. Consider Eq. (A7) for $H$. In

$$\delta H=\delta\hat{n}\cdot\underline{\Delta}\,\vec{r}+\hat{n}\cdot\delta(\underline{\Delta}\,\vec{r}),$$

the first term on the right–hand side vanishes since $\underline{\Delta}\,\vec{r}$ is parallel to $\hat{n}$. The variation of $\underline{\Delta}\,\vec{r}$, by simple arithmetic, is

$$\delta\underline{\Delta}\,\vec{r}=\delta\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)\vec{r}$$
$$=\left(\partial_i\frac{\delta\sqrt{g}}{\sqrt{g}}\right)\partial^i\vec{r}+\underline{\Delta}\,\delta\vec{r}+\frac{1}{\sqrt{g}}\partial_i[\sqrt{g}(\delta g^{ij})\partial_j]\vec{r}.$$

The first term in the last line, being proportional to $\partial^i\vec{r}$, vanishes once a scalar product with $\hat{n}$ is taken. For the same reason, the only contribution that survives in the last term is obtained when the derivative $\partial_i$ acts on $\vec{r}$. The variation $\delta g^{ij}$ is expressed in terms of $\delta\vec{r}$ by taking the variation of $g^{ik}g_{kj}=\delta^i_j$, so that finally

$$\delta H=\hat{n}\cdot\underline{\Delta}\,\delta\vec{r}-2(\partial^i\delta\vec{r}\partial^j\vec{r})(\hat{n}\cdot\partial_i\partial_j\vec{r}).$$

The variation of $\mathcal{H}_{c,1}=\kappa_1\int d^D s\sqrt{g}H$ with respect to $\delta\vec{r}$ involves the variation $\delta H$ and the variation of $\sqrt{g}$, which is evaluated as before. The functional derivative of the first term is evaluated from the above equation with a partial integration,

$$\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}_{c,1}}{\delta\vec{r}}=k_1\Big\{-H\underline{\Delta}\,\vec{r}-\partial^i\vec{r}\partial_i H+\underline{\Delta}\,\hat{n}$$
$$+2\frac{1}{\sqrt{g}}\partial_i[\sqrt{g}(\hat{n}\cdot\partial^i\partial_j\vec{r})\partial^j\vec{r}]\Big\}.$$

On multiplying the above equation by $\hat{n}$ to find $\mathcal{G}_{c,1}$, the property $\hat{n}\cdot\partial_i\vec{r}=0$ can be used again to show that the second term gives no contribution and the last term becomes $2b^i_j b^j_i$. This is twice the trace of the square of the matrix of the coefficients of the second fundamental form. Finally, we have to compute $\hat{n}\cdot\underline{\Delta}\,\hat{n}$. Using the facts that $\hat{n}\perp\partial^i\hat{n}$ and that $\partial_i\hat{n}=-b^j_i\partial_j\vec{r}$, we easily find

$$\hat{n}\cdot\underline{\Delta}\,\hat{n}=-\partial_i\hat{n}\cdot\partial^i\hat{n}=-b^j_i b^i_j=-\sum_{i=1}^{D}\lambda_i^2.\qquad\text{(D1)}$$

Collecting the various terms, we get the result displayed in Eq. (33), which is clearly fully $R$-invariant.

The chain rule of differentiation, applied to $\mathcal{H}_{c,p}=\kappa_p\int d^D s\sqrt{g}H^p$, also gives Eq. (35), where the second term, as before, comes from the variation of $\sqrt{g}$, while the others come from $\delta H^p=pH^{p-1}\delta H$. This also needs the straightforward generalization of Eq. (D1) to

$$\hat{n}\cdot\underline{\Delta}(F\hat{n})=\left(\underline{\Delta}-\sum_{i=1}^{D}\lambda_i^2\right)F$$

for a generic $R$-invariant function $F(\underline{s})$.

## APPENDIX E: ORIENTATIONAL ENERGY

The variation of Eq. (38) for a change $\vec{r}\to\vec{r}+\delta\vec{r}$ can be obtained as soon as we know $\delta\sqrt{g}$ and $\delta\hat{n}$. The former has already been obtained in Appendix C, and it is $\delta\sqrt{g}=\sqrt{g}g_{ij}\partial_i\vec{r}\cdot\partial_j\delta\vec{r}$, whereas the latter is derived as follows. Since $\delta\hat{n}\perp\hat{n}$ and

$$0=\delta(\hat{n}\cdot\partial_i\vec{r})=\partial_i\vec{r}\cdot\delta\hat{n}+\hat{n}\cdot\partial_i\delta\vec{r},$$

one has

$$\delta\hat{n}=-\partial^i\vec{r}(\hat{n}\cdot\partial_i\delta\vec{r}).\qquad\text{(E1)}$$

The variation of Eq. (38) is then

$$\delta\mathcal{H}_z=\int d^D s\sqrt{g}\Big[-g^{ij}(\partial_i\vec{r}\cdot\partial_j\delta\vec{r})\chi(n_z)$$
$$+\frac{d\chi(n_z)}{dn_z}(\hat{z}\cdot\partial^i\vec{r})(\hat{n}\cdot\partial_i\delta\vec{r})\Big],\qquad\text{(E2)}$$

from which it follows that

$$\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}_z}{\delta\vec{r}}=\frac{1}{\sqrt{g}}\partial_j\Big[\sqrt{g}g^{ij}\partial_i\vec{r}\chi(n_z)-\sqrt{g}\frac{d\chi(n_z)}{dn_z}\hat{n}(\hat{z}\cdot\partial^j\vec{r})\Big]$$
$$=\chi(n_z)\underline{\Delta}\,\vec{r}-\frac{d^2\chi(n_z)}{dn_z^2}\hat{n}(\hat{z}\cdot\partial^i\vec{r})\partial_i n_z$$
$$-\frac{d\chi(n_z)}{dn_z}\hat{n}(\hat{z}\cdot\underline{\Delta}\,\vec{r})$$
$$-\frac{d\chi(n_z)}{dn_z}(\partial^j\vec{r}\cdot\partial_j n_z-(\hat{z}\cdot\partial^j\vec{r})\partial_j\hat{n}).\qquad\text{(E3)}$$

The last two terms sum to zero since $\partial_i\hat{n}=-b^j_i\partial_j\vec{r}$ and from the symmetry of $b_{ij}$. Finally, using Eq. (A7) (i.e., $\underline{\Delta}\,\vec{r}=\hat{n}H$), we get

$$-\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}_z}{\delta\vec{r}}=\hat{n}G_z,\qquad\text{(E4)}$$

with

$$G_z = H\left( -\chi(n_z) + n_z \frac{d\chi(n_z)}{dn_z} \right) + \frac{d^2\chi(n_z)}{dn_z^2} \partial^i z \, \partial_i n_z \,, \tag{E5}$$

where $z \equiv \hat{z} \cdot \vec{r}$.

Using the Monge parametrization,

$$G_z = H\left( n_z \frac{d\chi(n_z)}{n_z} - \chi(n_z) \right) + \frac{d^2\chi(n_z)}{dn_z^2} \frac{\partial i h \, \partial_i g^{-1/2}}{g}, \tag{E6}$$

with $n_z = g^{-1/2}$ and $H = \vec{\nabla}(\vec{\nabla} h / \sqrt{g})$.

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