Dynamic Scaling of Growing Interfaces

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A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

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Many challenging problems are associated with growth patterns in clusters[1] and solidification fronts.[2] Several models have been proposed recently to describe the growth of smoke and colloid aggregates, flame fronts, tumors, etc.[1] It is generally recognized that the growth process occurs mainly at an "active" zone on the surface of the cluster, with interesting scaling properties.[3] However, a systematic *analytic* treatment of the static and dynamic fluctuations of the growing interface has been lacking so far.

In this paper we propose a model for the time evolution of the profile of a growing interface, and examine its properties. Guided by the ideas of universality we write down the simplest nonlinear, local differential equation governing the growth of the profile applicable to such processes as vapor deposition[4] or the Eden model.[5] The analysis of this equation is considerably simplified by mappings to two different, albeit more familiar, forms. One is the hydrodynamic problem of the Burgers's equation,[6] and the other is a directed polymer in a random environment.[7] The deterministic growth of the profile can in fact be obtained exactly, and its long-time relaxation behavior exhibits very interesting patterns related to the shock waves in Burgers's equation.[6] The stochastic growth is treated by dynamic renormalization-group techniques.[8] For a one-dimensional interface a fluctuation-dissipation theorem[9] exists, leading to an exact dynamic exponent $z=\frac{3}{2}$. This result is in excellent agreement with previous numerical simulations of ballistic aggregation[10] and Eden clusters.[11] For two-dimensional interfaces, the mapping to the random directed-polymer problem[7] is used to make an efficient indirect numerical simulation with the result $z \sim 1.5$. A nontrivial behavior is also predicted for the static fluctuations in this case.

The interface profile, suitably coarse-grained, is described by a height $h(\mathbf{x},t)$. As usual, it is convenient to ignore overhangs so that h is a single-valued function of $\mathbf{x}$. The simplest nonlinear Langevin equation for a local growth of the profile is given by[12]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x},t). \qquad (1)$$

The first term on the right-hand side describes relaxation of the interface by a surface tension ν. The second term is the lowest-order nonlinear term that can appear in the interface growth equation, and is justified later on with the Eden model as an example. Edwards and Wilkinson[13] have studied Eq. (1) without the nonlinear term, but we demonstrate that such a term is necessary, and responsible for the unusual properties of the growing interface. Higher-order terms can also be present, but they are irrelevant, and will not modify the universal scaling properties. The noise $\eta(\mathbf{x},t)$ has a Gaussian distribution with $\langle \eta(\mathbf{x},t)\rangle = 0$, and

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = 2D\,\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t'),$$

although the actual form of the distribution is irrelevant. In principle there is also a velocity term, which is removed by choice of an appropriate moving coordinate system. Note that Eq. (1) is invariant under translations $h \to h + \text{const}$, and obeys the infinitesimal reparametrization

$$h \to h + \boldsymbol{\epsilon}\cdot\mathbf{x}, \quad \mathbf{x} \to \mathbf{x} + \lambda\boldsymbol{\epsilon}t,$$

which describes the tilting of the interface by a small angle.

To justify the nonlinear term in Eq. (1), consider the growth of an Eden cluster[5] taking place by addition of particles to the surface. As indicated in Fig. 1, growth occurs in a direction locally normal to the interface. When a particle is added, the increment projected along the h axis (Fig. 1 inset) is $\delta h = [(v\delta t)^2 + (v\delta t \nabla h)^2]^{1/2}$, resulting in

$$\dot{h} = v[1 + (\nabla h)^2]^{1/2} \simeq v + (v/2)(\nabla h)^2 + \dots .$$

After transformation to the comoving frame, and inclusion of a surface tension (obtained if surface particles are allowed to diffuse and relax), the original growth equation is regained. Such a nonlinear term is clearly expected in all situations where lateral growth is allowed.[14]

Equation (1) can be mapped to two other useful and possibly more familiar forms. Following the transformation $W(\mathbf{x},t) = \exp[(\lambda/2\nu)h(\mathbf{x},t)]$, we obtain

$$\partial W/\partial t = \nu \nabla^2 W + (\lambda/2\nu)\eta(\mathbf{x},t) W, \tag{2}$$

which is a diffusion equation in a time-dependent random potential. In fact $W(\mathbf{x},t)$ can be regarded as the sum of Boltzmann weights for all static configurations of a directed polymer in a $(d+1)$-dimensional space from $(\mathbf{0},0)$ to $(\mathbf{x},t)$. The noise term then describes a quenched random potential $(\lambda/2\nu)\eta(\mathbf{x},t)$ exerted by the environment on the polymer. The second transformation, $\mathbf{v} = -\nabla h$, results in

$$\partial \mathbf{v}/\partial t + \lambda \mathbf{v} \cdot \nabla \mathbf{v} = \nu \nabla^2 \mathbf{v} - \nabla \eta(\mathbf{x},t), \tag{3}$$

which is (for $\lambda = 1$) the Burgers's equation for a vorticity-free velocity field.[6] [The connection between Eqs. (2) and (3) has already been noted.[15]] Both transformations provide valuable insight into the interface problem.

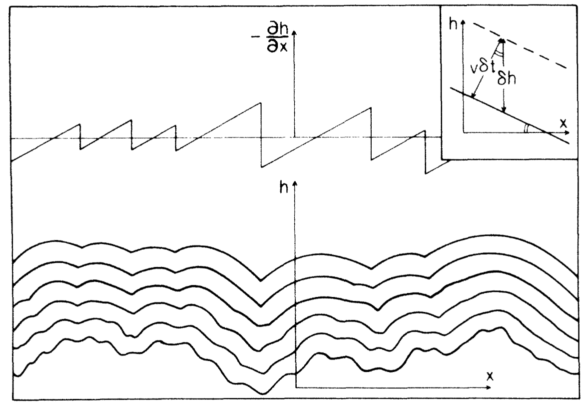

FIG. 1. Successive profiles for a deterministic growth by a process similar to Eq. (1). Inset indicates how growth occurs locally along the normal to the interface. The gradient of the profile develops "shock waves" as explained in the text.

There is extensive literature on the study of dendrite formation, and pattern selection through complicated deterministic models.[2] The deterministic version of Eq. (1), i.e., with $\eta(\mathbf{x},t) = 0$, does indeed possess uniformly moving solutions resembling dendrites, such as

$$h(\mathbf{x},t) = (2\nu/\lambda)[\ln|\cos(\mathbf{k}\cdot\mathbf{x})| - \nu k^2 t].$$

However, these solutions are inherently unstable, and a typical initial condition leads to an asymptotically smooth interface. The relaxation pattern in this case is still interesting, and very different from the ordinary surface-tension–dominated case. In the absence of noise Eq. (2) can be solved exactly subject to any initial condition. If the initial profile is $h(\mathbf{x},0) = h_0(\mathbf{x})$, its evolution is given by

$$h(\mathbf{x},t) = \frac{2\nu}{\lambda} \ln\left\{\int_{-\infty}^{\infty} \frac{d^d\xi}{(4\pi\nu t)^{d/2}} \exp\left[-\frac{(\mathbf{x}-\boldsymbol{\xi})^2}{4\nu t} + \frac{\lambda}{2\nu} h_0(\boldsymbol{\xi})\right]\right\}. \tag{4}$$

The long-time behavior is obtained by methods similar to those used for Burgers's equation.[6] The asymptotic form of the solution is composed of paraboloid segments $h_n = A_n - (\mathbf{x} - \boldsymbol{\xi}_n)^2/2\lambda t$, joined together with discontinuities in ∇h. It can be proved, by the generalization of results of Ref. 6, that if the initial interface is rough {i.e., $P(h_0(x)) \sim \exp[-\int dx (\nabla h)^2]$}, the average size of these paraboloids grows in time as $t^{2/3}$ in one dimension. A typical one-dimensional growth pattern is sketched in Fig. 1, together with the asymptotic form of $v = -\partial h/\partial x$. The relation between the parabolic segments and the shock waves of Burgers's equation[6] is apparent from this figure. Further evolution of the pattern proceeds through the larger parabolas' growing at the expense of the smaller ones, and parallels the evolution of shock waves which is perhaps more familiar. The pattern has similarities to geological stratifications and successive layers of snow drifts. We thus have an intriguing connection between evolutions of a hydrodynamic and a growth pattern!

Let us now return to the full stochastic problem defined by Eq. (1). The formalism of the dynamic renormalization group[8] can be applied to a study of the scaling of time-dependent fluctuations. Indeed, this procedure has already been applied by Forster, Nelson, and Stephen[16] to Burgers's equation [Eq. (3)], and their results can be directly taken over. However, since we believe this technique to be important for more complex interface problems, it is briefly outlined here for completeness. The spatial Fourier transform of Eq. (1) is rewritten as

$$h(\mathbf{k},t)=G_0(\mathbf{k},t)h(\mathbf{k},0)+\int_0^t d\tau\, G_0(\mathbf{k},t-\tau)\left[\eta(\mathbf{k},\tau)-\frac{\lambda}{2}\int\frac{d^dq}{(2\pi)^d}\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})h(\mathbf{q},\tau)h(\mathbf{k}-\mathbf{q},\tau)\right].$$

The integral equation is then solved perturbatively in the vertex $-(\lambda/2)\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})$, using the free propagator $G_0(\mathbf{k},t)=\exp(-\nu k^2 t)\theta(t)$, and

$$\langle\eta(\mathbf{k},t)\eta(\mathbf{k}',t')\rangle=2D\delta^d(\mathbf{k}+\mathbf{k}')\delta(t-t').$$

The first terms in the perturbation expansions for the full propagator and for $\langle h(\mathbf{k},t)h(-\mathbf{k},t)\rangle$ diverge for $d<2$. The perturbation series is then reorganized into a renormalization-group calculation by only integrating out the modes with $e^{-l}\Lambda<|\mathbf{k}|<\Lambda$. The parameters are then rescaled as $k'=e^l\mathbf{k}$, $t'=e^{-zl}t$, and the remaining modes as $h'(\mathbf{k}',t')=e^{-(d+\chi)l}h(\mathbf{k},t)$. The rescaled modes then obey Eq. (1) with renormalized coefficients that to the lowest order are

$$d\nu/dl=[z-2+K_d\bar\lambda^2(2-d)/4d]\nu,$$
$$dD/dl=[z-d-2\chi+K_d\bar\lambda^2/4]D, \tag{5a}$$
$$d\lambda/dl=[\chi+z-2]\lambda,$$

with $K_d=S_d/(2\pi)^d$ and $\bar\lambda^2=\lambda^2D/\nu^3$. (The diagrams contributing to $d\lambda/dl$ cancel at this order.) The exponents z and χ are adjusted so that $d\nu/dl=dD/dl=0$. The last equation then indicates that the effective coupling constant $\bar\lambda$ evolves under rescaling as

$$\frac{d\bar\lambda}{dl}=\frac{2-d}{2}\bar\lambda+K_d\frac{(2d-3)}{4d}\bar\lambda^3. \tag{5b}$$

That this result is identical to the one for Burgers's equation[16] is, of course, no surprise. The exponents χ and z (evaluated at a fixed point) completely describe the scaling of the interface. For example, the asymptotic behavior of interface width in a strip geometry is $w(L,t)=L^\chi w_0(t/L^z)$. The exponent χ is related to the conventional hydrodynamic exponent η

$$[\langle\mathbf{v}(\mathbf{k},\omega)\mathbf{v}(-\mathbf{k},-\omega)\rangle=k^{\eta-2}g(\omega/k^z)]$$

through

$$\chi=(2-d)/2+(2-\eta-z)/2. \tag{6}$$

In the absence of nonlinearities the "ideal" exponent is $\chi=(2-d)/2$, which is the same as in roughening models ($z=2$ in this case).[14] In general, however, χ is different from the roughening exponent. For a cluster growing in a spherical geometry with a radius $R\sim t$, the interface width grows as $w(R,t)\sim R^\chi w_0(t/R^z)\sim t^{\chi/z}$. If for N particles $R\sim N^\nu$, then $w\sim N^{\nu'}$, and $\nu'=\chi\nu/z$ is the exponent originally postulated by Plischke and Rácz[3] for the width of the active zone.

The expected scaling behavior is now explored in various numbers of dimensions.

(a) *d=1.*—The expansion results $z=\frac{3}{2}$ and $\chi=\frac{1}{2}$ are actually exact. This is a consequence of a fluctuation-dissipation theorem[9] that holds in $d=1$ only, and ensures $\eta=2-z$; i.e., ν and D scale the same way to all orders in perturbation theory, as is evident to the lowest order from Eqs. (5a). Another consequence of this theorem is that χ [Eq. (6)] takes the same value as for an "ideal" interface. Since such a theorem does not hold for $d\neq1$, χ is in general "nonideal," notwithstanding recent conjectures to the contrary.[17,18] The dynamic scaling form has in fact been confirmed by the numerical studies of Plischke and Rácz[11] on the Eden model. They find $z=1.55\pm0.15$, in excellent agreement with the above prediction of $\frac{3}{2}$. There are also a number of simulations confirming that for a strip geometry the width scales as L^χ with $\chi=\frac{1}{2}$.[10,11,17,18] This was interpreted[17,18] as indicating an "ideal" interface, but as discussed above, it is a result of a peculiarity of one-dimensional interfaces.

Further support for the universal character of these exponents is provided by the earlier numerical results of Family and Viscek[10] on a "ballistic deposition model," which is a realistic description of vapor deposition processes.[4] They observe that initially the interface width grows with time (t is proportional to M, the mean number of deposited particles in their notation) as $t^{0.30\pm0.02}$, very close to our prediction of $\nu'=\chi/z=\frac{1}{3}$. Eventually the width scales with the sample size L as L^χ with $\chi=0.42\pm0.02$, not very far from 0.5.

(b) *d=2.*—This is the critical dimension of the model. However, Eq. (5b) indicates that the coupling $\bar\lambda$ is marginally relevant, and grows under rescaling. The fixed point determining the strong-coupling behavior is not accessible by perturbation theory. Also, since λ is only marginally relevant, direct numerical simulations[17,18] may be hampered by a large crossover regime, before the true asymptotic scaling is observed. Here the mapping to the directed-polymer problem [Eq. (2)] becomes helpful. At zero temperature, the directed-polymer problem maps directly to $\lambda\to\infty$, and the strong-coupling behavior can thus be probed. In fact, searching for the optimal polymer configuration at zero temperature is computationally much faster than summing over all configurations at finite temperature.[7] Preliminary numerical studies of the polymer problem in $d=2$ indicate a nontrivial behavior with $1/z=0.62\pm0.04$ and $\chi/z=0.33\pm0.03$. Although the results are not quite conclusive, it appears as though $z\sim\frac{3}{2}$ and $\chi\sim\frac{1}{2}$ may be regained, in which case these exponents will be superuniversal (independent of dimension). More numerical and experimental studies of this problem are certainly welcome.

(c) *$d \geqslant 3$.*—For dimensionalities larger than 2 the coupling λ is irrelevant, and an asymptotically ideal smooth surface is expected with $z=2$. In principle, Eq. (5b) allows for the possibility of a phase transition to different scaling for strong enough couplings. This is probably not relevant to the growth problem, and will be discussed more extensively in connection with the directed polymer.[7]

In conclusion, we have described how a growing interface can be studied by dynamic renormalization techniques.[8] This allows a classification of growth processes by their universality. The simplest possible interface model allowing for both surface tension and lateral growth is proposed in Eq. (1) The results obtained from this model are in excellent agreement with numerical simulations of the Eden model[11] and other growth processes.[10] In fact, Eq. (1) embodies three different universality classes. With $\lambda=\nu=0$, it corresponds to the random deposition model[19] with a diffusive interface ($h \sim t^{1/2}$). With $\lambda=0$, it corresponds to an ideal interface[13] with $\chi=(2-d)/2$ and $z=2$. For $\lambda \neq 0$ we obtained a new universality class with $\chi=\frac{1}{2}$ and $z=\frac{2}{3}$ (possibly independent of d). These results provide a framework for experimental measurements of χ and z in growth processes. The theoretical task is to generalize the results to more complicated growth mechanisms, with nonlocal interactions[20] or with a diffusive field, which may be more relevant to solidification fronts,[2] and thus to obtain a more complete picture of possible universality classes of growing interfaces.

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[14] Two further remarks regarding Eq. (1): (a) Some caution is necessary in discussing Eden processes and ballistic aggregation. Unlike more realistic growth processes, these models do not allow relaxation and hence $V=0$, which is a rather singular limit of Eq. (1). Comparison with simulations suggests that this is not a serious problem (at least in $d=1$). (b) This equation also describes corrosion processes, but with $\lambda<0$. However, as the results obtained in this paper are independent of the sign of λ, they should also describe a corroding interface (an "acid rain model").

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