Stochastic model of hysteresis

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The methods of the probability theory have been used in order to build up a model of hysteresis which is different from the well-known Preisach model. It is assumed that the system consists of large number of abstract particles in which the variation of an external control parameter (e.g., the magnetic field) may result in transitions between two states $\mathcal{S}^{(+)}$ and $\mathcal{S}^{(-)}$. The state of a particle is characterized by the value $+1$ or -1 of a random variable (e.g., the magnetization direction parallel or antiparallel to the magnetic field). The transitions are governed by two further random variables corresponding to the $\mathcal{S}^{(-)}\Rightarrow\mathcal{S}^{(+)}$ and the $\mathcal{S}^{(+)}\Rightarrow\mathcal{S}^{(-)}$ transitions (e.g., ''up switching'' and ''down switching magnetic field''). The method presented here makes it possible to calculate the probability distribution and consequently the expectation value of the number of particles in the $\mathcal{S}^{(+)}$ (or $\mathcal{S}^{(-)}$) state for both increasing and decreasing parameter values, i.e., the hysteresis curves of the transitions can be determined. It turns out that the reversal points of the control parameter are Markov points which determine the stochastic evolution of the process. It has been shown that the branches of the hysteresis loop are converging to fixed limit curves when the number of cyclic back-and-forth variations of the control parameter between two consecutive reversal points is large enough. This convergence to limit curves gives a clear explanation of the accommodation process. The accommodated minor loops show the return-point memory property but this property is obviously absent in the case of nonaccommodated minor loops which are not congruent and generally not closed. In contrast to the traditional Preisach model the reversal point susceptibilities are nonzero finite values. The stochastic model can provide a surprisingly good approximation of the Raylaigh quadratic law when the external parameter varies between two sufficiently small values. The practical benefits of the model can be seen in the numerical analysis of the derived equations. On one hand the calculated curves are in good qualitative agreement with the experimental observations and on the other hand, the estimation of the joint distribution function of the up and down switching fields can be performed by using the measured hysteresis curves.

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I. INTRODUCTION

The phenomenon of the hysteresis, understood in a general sense, has been investigated so intensively for many decades that any list of references would be far from complete by any standards. It is very fortunate that in the last few years outstanding monographs [1–4] have been published in this field, and thus the author does not feel obliged to cite the large amount of old but important references. However, it is considered important to mention two papers of Kádár [5,6] whose work has played a stimulating role in getting to the idea of reconsideration of the hysteresis theory by the present author. There is no doubt that the abstract reformulation of the Preisach model [7] given by Krasnoselskii and Pokrovskii [8] and summarized by Mayergoyz [1] in his book resulted in an improved mathematical clarity in the hysteresis theory, but the stochastic nature of the hysteresis still has not been treated with sufficient mathematical rigor [16].

The aim of the present paper is to define a stochastic model of hysteresis and to derive exact equations for the probability distribution functions describing the state variations in hysteretic processes as a function of increasing as well as decreasing control parameters. The vocabulary of magnetic hysteresis will be used for convenience from now on, however, the concepts can easily be generalized for any hysteretic phenomenon.

II. DESCRIPTION OF THE MODEL

Let us assume that the unit volume of the system consists of many small abstract regions, called ''particles'' which are characterized by four random variables μ,λ,χ_d, and χ_u. The absolute value of the particle magnetization is denoted by μ. If the particle magnetization is parallel (antiparallel) to the external magnetic field H then the particle is in the state $\mathcal{S}^{(+)}(\mathcal{S}^{(-)})$ and $\lambda=1(-1)$. The random value χ_d corresponds to a local field at which the state $\mathcal{S}^{(+)}$ jumps to the state $\mathcal{S}^{(-)}$ and similarly the $\mathcal{S}^{(-)}\Rightarrow\mathcal{S}^{(+)}$ transition occurs at the random local field χ_u. For simplicity the χ_u and χ_d will be called the U and D fields. These two random variables express the obvious fact that each particle ''feels'' not only the external magnetic field H, but also the interaction field due to the adjacent particles and the random field originated from the inhomogenities of the surrounding environment. These particles characterized by the random variables μ,λ,χ_d, and χ_u can be regarded as ''independent'' abstract elements of the system, and they will be called ''*hysterons*.''

Figure 1 illustrates a possible realization of transitions $\mathcal{S}^{(+)}\Leftrightarrow\mathcal{S}^{(-)}$ of a hysteron. The transition curves form a random rectangular hysteresis loop which is almost in all cases asymmetrical in the coordinate system of magnetization versus external magnetic field since the U and D fields are supposed to be random.

Let us denote the hysterons in a system by $h_1,h_2,\ldots,h_N$, and let $\mathcal{N}^{(+)}$ be the set of indices and $n^{(+)}$

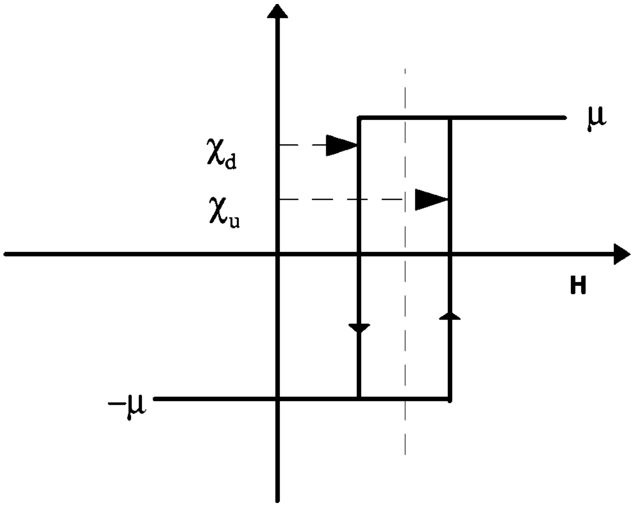

FIG. 1. A possible realization of the transition $\mathcal{S}^{(+)} \Leftrightarrow \mathcal{S}^{(-)}$.

the number of hysterons which are in the state $\mathcal{S}^{(+)}$ at a given external field H. In this case the magnetization of the system is given by the stochastic equation

$$\delta_{n^{(+)}} = \sum_{k=1}^{N} \lambda_k \mu_k ,$$

where μ_k is the absolute value of the magnetization of the hysteron h_k, while

$$\lambda_k = \begin{cases} 1, & \text{if } k \in \mathcal{N}^{(+)}, \\ -1, & \text{if } k \ni \mathcal{N}^{(+)}. \end{cases}$$

Since the random variables $\mu_1, \mu_2, \ldots, \mu_N$ are mutually independent and have the same probability distribution function

$$\mathbf{P}\{\mu_k \leqslant x\} = L(x), \ \forall k = 1,2, \ldots, N,$$

it is obvious that the characteristic function of the distribution function

$$\mathbf{P}\{\delta_{n^{(+)}} \leqslant x\} = R_{n^{(+)}}(x) \tag{1}$$

can be written in the form

$$\Phi_{n^{(+)}}(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} dR_{n^{(+)}}(x) = [\varphi(-\omega)]^N \left[\frac{\varphi(\omega)}{\varphi(-\omega)}\right]^{n^{(+)}}, \tag{2}$$

where

$$\varphi(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} dL(x) = \int_{0}^{+\infty} e^{i\omega x} dL(x). \tag{3}$$

In order to calculate the characteristic function

$$\Phi(H,\omega) = \sum_{n^{(+)}=0}^{N} \Phi_{n^{(+)}}(\omega) p_{n^{(+)}}(H)$$
$$= [\varphi(-\omega)]^N \sum_{n^{(+)}=0}^{N} p_{n^{(+)}}(H) \left[\frac{\varphi(\omega)}{\varphi(-\omega)}\right]^{n^{(+)}}, \tag{4}$$

we need the probability of finding $n^{(+)}$ hysterons in the state $\mathcal{S}^{(+)}$ at the external field H which is the endpoint of a well-defined magnetization prehistory. The determination of this probability and the derivation of the equations for ‘‘up’’ and ‘‘down’’ magnetizations versus magnetic field will be the task of the next section.

III. DERIVATION OF THE FUNDAMENTAL EQUATIONS

A. Some basic relations

Let us denote by $H(x,y|\mathcal{C})$ the joint distribution function of the random U and D fields. From the physical point of view it is quite obvious that the U field cannot be smaller than the D field, so the stochastic inequality $\chi_u \geqslant \chi_d$ must be satisfied. It is easy to show [9] that the joint distribution function of χ_u and χ_d satisfying the condition $\mathcal{C} = \{\chi_u \geqslant \chi_d\}$ can be written in the form

$$\mathbf{P}\{\chi_u \leqslant x, \chi_d \leqslant y | \chi_u \geqslant \chi_d\}$$
$$= H(x,y|\mathcal{C})$$
$$= \frac{\int_{-\infty}^{x} dx' \int_{-\infty}^{y} h(x',y') \Delta(x'-y') dy'}{\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{x'} h(x',y') dy'}, \tag{5}$$

where $\Delta(x)$ is the unit step function. It is clear that the joint density function of the U and D fields can be given by

$$h(x,y|\mathcal{C}) = \frac{h(x,y)}{\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{x'} h(x',y') dy'} \Delta(x-y), \tag{6}$$

provided that the condition $\mathcal{C}$ is valid.

We need in the sequel two conditional probability distribution functions

$$\mathbf{P}\{\chi_u \leqslant x | \mathcal{C}\} = H(x,\infty|\mathcal{C}) = F_u(x|\mathcal{C})$$
$$= \frac{\int_{-\infty}^{x} dx' \int_{-\infty}^{x'} h(x',y') dy'}{\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{x'} h(x',y') dy'} \tag{7}$$

and

$$\mathbf{P}\{\chi_d \leqslant y|\mathcal{C}\} = H(\infty,y|\mathcal{C}) = F_d(y|\mathcal{C})$$
$$= \frac{\int_{-\infty}^{y} dy' \int_{y'}^{+\infty} h(x',y')dx'}{\int_{-\infty}^{+\infty} dy' \int_{y'}^{+\infty} h(x',y')dx'}. \quad (8)$$

Evidently $F_u(x|\mathcal{C})$ is the probability that the U field of a given hysteron is not larger than x, while $F_d(y|\mathcal{C})$ is the probability that the D field is not larger than y assuming in both cases that the condition $\mathcal{C}$ is fulfilled. By using the Dirichlet's theorem for changing the sequence of integration it is obvious that

$$\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{x'} h(x',y')dy' = \int_{-\infty}^{+\infty} dy' \int_{y'}^{+\infty} h(x',y')dx'.$$

For the sake of further considerations it is necesarry to introduce two *transition probabilities* denoted by $w_u(H_l \uparrow H)$ and $w_d(H_u \downarrow H)$. Let H_l be a fixed value of the external magnetic field and let us suppose that the state of a given hysteron is $\mathcal{S}^{(-)}$ at H_l. By using elementary theorems, it can be proved that if the external field increases monotonically from H_l to $H \geqslant H_l$ then

$$w_u(H_l \uparrow H) = \frac{\int_{H_l}^{H} dx \int_{-\infty}^{x} h(x,y)dy}{\int_{H_l}^{+\infty} dx \int_{-\infty}^{x} h(x,y)dy} \quad (9)$$

is the probability of the transition $\mathcal{S}^{(-)} \Rightarrow \mathcal{S}^{(+)}$ occuring in the interval $[H_l \uparrow H]$. Similarly, let H_u be an other fixed value and $\mathcal{S}^{(+)}$ the state of a hysteron at H_u. If the external field decreases now monotonically from H_u to $H \leqslant H_u$ then

$$w_d(H_u \downarrow H) = \frac{\int_{H}^{H_u} dy \int_{y}^{+\infty} h(x,y)dx}{\int_{-\infty}^{H_u} dy \int_{y}^{+\infty} h(x,y)dx} \quad (10)$$

gives the probability of the transition $\mathcal{S}^{(+)} \Rightarrow \mathcal{S}^{(-)}$ occuring in the interval $[H_u \downarrow H]$.

B. Stochastic magnetizing process

Let us introduce the ''time parameter'' $t \in [0,+\infty]$ and define a real valued, external field function $H(t)$ which consists of monotone increasing and decreasing sections of different length. Denote by $H_1, H_2, \ldots, H_j, \ldots$, the extremum values of the function $H(t)$ belonging to the subsequent time points $t_1 < t_2 < \cdots < t_j < \cdots$. It is clear that if $H(t_j) = H_j$ is a local maximum then $H(t_{j-1}) = H_{j-1}$ and $H(t_{j+1}) = H_{j+1}$ must be local minimums which are not necessarily equal. In the following the sequence $\{H_j\}$ will be called *magnetizing path* and the elements of this sequence are called *points of reversal*. If the functions $H^{(1)}(t)$ and $H^{(2)}(t)$ have the same magnetizing path then they are said to be equivalent for any magnetizing process irrespective of the form of the time function between the individual extrema. In Fig. 2 two *equivalent* $H(t)$ *functions* are seen. The sequence of extrema is the same for both curves, but the time distance and the shape of sections between the consecutive extrema are different.

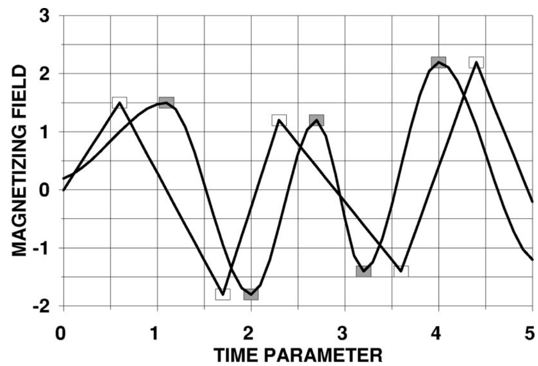

FIG. 2. Two equivalent $H(t)$ curves.

It is assumed that the magnetizing process which consists of random transitions $\mathcal{S}^{(+)} \Leftrightarrow \mathcal{S}^{(-)}$ of hysterons does not ''feel'' the variation speed of $H(t)$ between the consecutive extremum values, i.e., the magnetizing process is *static*. The evolution of the process in each subinterval $[H_j, H_{j+1}]$, $j = 1,2, \ldots$, is stochasically determined by the extremum H_j and by the actual values of $H(t)$ following H_j, but the process does not depend on the time derivative of $H(t)$. This property is called *rate independence* in the nonstochastic theory of hysteresis [2] but it will be applied in this stochastic theory too. Denote the maximum reversal fields by odd and the minimum ones by even indices. In this case it is clear that

$$H_{2k-1} \geqslant H_{2k} \leqslant H_{2k+1},$$

and naturally any one of the inequalities $H_{2k+1} \geqslant H_{2k-1}$ and $H_{2k+1} \leqslant H_{2k-1}$ can be valid.

Now, let us define *the random function* $\xi_d^{(+)}(H_{2k-1} \downarrow H)$ which gives the number of hysterons in the state $\mathcal{S}^{(+)}$ at the decreasing external field H belonging to the interval $[H_{2k-1} \downarrow H_{2k}]$. Similarly, denote by $\xi_u^{(+)}(H_{2k} \uparrow H)$ the number of hysterons in the state $\mathcal{S}^{(+)}$ at the increasing external field H belonging to the interval $[H_{2k} \uparrow H_{2k+1}]$.

We suppose that at the starting point of the magnetizing process each hysteron is in the $\mathcal{S}^{(-)}$ state, that is the system is in the state of negative saturation. In the following this fact will be expressed by the stochastic equation $\xi_{\rm start}^{(+)} = 0$. Evidently any other state of the system could as well be chosen for the starting point, this choice, however, does not really matter since the influence of the starting state on the evolution of the process — as it will be shown — disappears very rapidly.

In order to describe the magnetizing process we should determine two probabilities. One of them is

$$\mathbf{P}\{\xi_d^{(+)}(H_{2k+1} \downarrow H) = n_{2k+1}^{(d)}(H) | \xi_{start}^{(+)} = 0\}$$
$$= p_{2k+1}^{(d)}[H_{2k+1} \downarrow H, n_{2k+1}^{(d)}(H) | 0] \quad (11)$$

and the other is

$$\mathbf{P}\{\xi_u^{(+)}(H_{2k}\uparrow H)=n_{2k}^{(u)}(H)|\xi_{start}^{(+)}=0\}$$
$$=p_{2k}^{(u)}[H_{2k}\uparrow H,n_{2k}^{(u)}(H)|0]. \quad (12)$$

It is important to note that the reversal points (extremum values)

$$H_1,H_2,\ldots,H_{2k-1},H_{2k},H_{2k+1},\ldots,$$

are *Markov points of the stochastic processes* $\xi_u^{(+)}(H_j\uparrow H)$ and $\xi_d^{(+)}(H_{j+1}\downarrow H)$, and therefore we can write the following equations:

$$p_{2k+1}^{(d)}[H_{2k+1}\downarrow H,n_{2k+1}^{(d)}(H)|0]$$
$$=\sum_{n_{2k}^{(u)}(H_{2k+1})=0}^{N} p_{2k+1}^{(d)}$$
$$\times[H_{2k+1}\downarrow H,n_{2k+1}^{(d)}(H)|n_{2k}^{(u)}(H_{2k+1})]p_{2k}^{(u)}[H_{2k}\uparrow$$
$$\times H_{2k+1},n_{2k}^{(u)}(H_{2k+1})|0] \quad (13)$$

and

$$p_{2k}^{(u)}[H_{2k}\uparrow H,n_{2k}^{(u)}(H)|0]$$
$$=\sum_{n_{2k-1}^{(d)}(H_{2k})=0}^{N} p_{2k}^{(u)}$$
$$\times[H_{2k}\uparrow H,n_{2k}^{(u)}(H)|n_{2k-1}^{(d)}(H_{2k})]p_{2k-1}^{(d)}$$
$$\times[H_{2k-1}\downarrow H_{2k},n_{2k-1}^{(d)}(H_{2k})|0]. \quad (14)$$

As it has already been mentioned the hysterons can be regarded as independent of each other particles and, therefore, it is an easy task to determine the probability that the number of $\mathcal{S}^{(+)}$ hysterons is exactly equal to a non-negative integer not larger than *N*, at an either decreasing or increasing external field *H* provided that the number of $\mathcal{S}^{(+)}$ hysterons is known at the last reversal point before arriving at *H*.

The probability $p_{2k+1}^{(d)}[H_{2k+1}\downarrow H,n_{2k+1}^{(d)}(H)|n_{2k}^{(u)}(H_{2k+1})]$ can be obtained as a result of the following consideration. If the number of $\mathcal{S}^{(+)}$ hysterons at the reversal point H_{2k+1} is equal to $n_{2k}^{(u)}(H_{2k+1})$, then — in order to have $n_{2k+1}^{(d)}(H)$ hysterons in the state $\mathcal{S}^{(+)}$ at the external field $H\leqslant H_{2k+1}$ — exactly $n_{2k}^{(u)}(H_{2k+1})-n_{2k+1}^{(d)}(H)$ hysterons of state $\mathcal{S}^{(+)}$ have to transform to the state $\mathcal{S}^{(-)}$ in the interval $[H_{2k+1}\downarrow H]$. It is obvious that the probability of this event can be given by

$$p_{2k+1}^{(d)}[H_{2k+1}\downarrow H,n_{2k+1}^{(d)}(H)|n_{2k}^{(u)}(H_{2k+1})]$$
$$=\binom{n_{2k}^{(u)}(H_{2k+1})}{n_{2k+1}^{(d)}(H)}[w_d(H_{2k+1}\downarrow H)]^{n_{2k}^{(u)}(H_{2k+1})-n_{2k+1}^{(d)}(H)}$$
$$\times[1-w_d(H_{2k+1}\downarrow H)]^{n_{2k+1}^{(d)}(H)}. \quad (15)$$

Similarly, to determine the probability $p_{2k}^{(u)}[H_{2k}\uparrow H,n_{2k}^{(u)}(H)|n_{2k-1}^{(d)}(H_{2k})]$ one has to recognize that if the number of $\mathcal{S}^{(-)}$ hysterons at the reversal point H_{2k} is equal to $N-n_{2k-1}^{(d)}(H_{2k})$, then — in order to have $n_{2k}^{(u)}(H)$ hysterons in the state $\mathcal{S}^{(+)}$ at the external field $H\geqslant H_{2k}$ — exactly $n_{2k}^{(u)}(H)-n_{2k-1}^{(d)}(H_{2k-1})$ hysterons of state $\mathcal{S}^{(-)}$ have to transform to the state $\mathcal{S}^{(+)}$ in the interval $[H_{2k}\uparrow H]$. The probability of this event is given by

$$p_{2k}^{(u)}[H_{2k}\uparrow H,n_{2k}^{(u)}(H)|n_{2k-1}^{(d)}(H_{2k})]$$
$$=\binom{N-n_{2k-1}^{(d)}(H_{2k})}{n_{2k}^{(u)}(H)-n_{2k-1}^{(d)}(H_{2k})}$$
$$\times[w_u(H_{2k}\uparrow H)]^{n_{2k}^{(u)}(H)-n_{2k-1}^{(d)}(H_{2k})}$$
$$[1\quad -w_u(H_{2k}\uparrow H)]^{N-n_{2k}^{(u)}(H)}. \quad (16)$$

In order to symplify the further calculations let us introduce *the generating functions*

$$\Gamma_{2k+1}^{(d)}(H_{2k+1}\downarrow H,z)$$
$$=\sum_{n_{2k+1}^{(d)}(H)=0}^{N} p_{2k+1}^{(d)}[H_{2k+1}\downarrow H,n_{2k+1}^{(d)}(H)|0]z^{n_{2k+1}^{(d)}(H)} \quad (17)$$

and

$$\Gamma_{2k}^{(u)}(H_{2k}\uparrow H,z)=\sum_{n_{2k}^{(u)}(H)=0}^{N} p_{2k}^{(u)}[H_{2k}\uparrow H,n_{2k}^{(u)}(H)|0]z^{n_{2k}^{(u)}(H)}. \quad (18)$$

By using the Eqs. (13) and (15) we get *the first fundamental equation* in the form

$$\Gamma_{2k+1}^{(d)}(H_{2k+1}\downarrow H,z)=\Gamma_{2k}^{(u)}[H_{2k}\uparrow H_{2k+1},a(H_{2k+1},H,z)], \quad (19)$$

where

$$a(H_{2k+1},H,z)=w_d(H_{2k+1}\downarrow H)+[1-w_d(H_{2k+1}\downarrow H)]z. \quad (20)$$

The second fundamental equation follows from the relations (14) and (16). We have

$$\Gamma_{2k}^{(u)}(H_{2k}\uparrow H,z)=[c(H_{2k},H,z)]^N\Gamma_{2k-1}^{(d)}$$
$$\times[H_{2k-1}\downarrow H_{2k},b(H_{2k},H,z)], \quad (21)$$

where

$$c(H_{2k},H,z)=1-(1-z)w_u(H_{2k}\uparrow H) \quad (22)$$

and

$$b(H_{2k},H,z)=\frac{z}{c(H_{2k},H,z)}. \quad (23)$$

Now we will derive the characteristic function of the probability that the system magnetization is not larger than *x* at a decreasing external field *H* which follows the last reversal point H_{2k+1}. Introducing the notation

$$\psi(\omega)=\frac{\varphi(\omega)}{\varphi(-\omega)} \quad (24)$$

and by using the relation (17) we obtain from Eq. (4)

$$\Phi_d(H_{2k+1}\downarrow H,\omega)=[\varphi(-\omega)]^N\Gamma_{2k+1}^{(d)}[H_{2k+1}\downarrow H,\psi(\omega)]. \tag{25}$$

Similarly, if we use Eq. (18) then the characteristic function of the probability that the magnetization of the system is not larger than x at an increasing external field H after the last reversal point H_{2k} can be obtained from Eq. (4) in the form

$$\Phi_u(H_{2k}\uparrow H,\omega)=[\varphi(-\omega)]^N\Gamma_{2k}^{(u)}[H_{2k}\uparrow H,\psi(\omega)]. \tag{26}$$

These two characteristic functions describe completely the stochastic behavior of the magnetizing process in both increasing and decreasing external magnetic fields. It is apparent from the above considerations that the stochastic model developed by us has been built up without any reference to a particular nature of hysteresis and therefore, its generality is at least as high as that of the Krasnoselskii and Pokrovskii [8] model.

C. Calculation of the hysteresis curves

The expectation value of the magnetic moment μ_k due to the kth hysteron can be given by

$$\mathbf{E}\{\mu_k\}=i^{-1}\left[\frac{d\varphi(\omega)}{d\omega}\right]_{\omega=0}=M_s .$$

By using this expression we can write the expectation value of the magnetization of the system at the decreasing external field H following the reversal point H_{2k+1} in the form

$$\begin{aligned} i^{-1}&\left[\frac{d\Phi_d(H_{2k+1}\downarrow H,\omega)}{d\omega}\right]_{\omega=0}\\ &=M_{2k+1}^{(d)}(H_{2k+1}\downarrow H|0)\\ &=2M_sN_{2k+1}^{(d)}(H_{2k+1}\downarrow H|0)-NM_s, \end{aligned} \tag{27}$$

where

$$N_{2k+1}^{(d)}(H_{2k+1}\downarrow H|0)=\left[\frac{d\Gamma_{2k+1}^{(d)}(H_{2k+1}\downarrow H,z)}{dz}\right]_{z=1}. \tag{28}$$

From the fundamental Eq. (19) we obtain

$$\begin{aligned} &N_{2k+1}^{(d)}(H_{2k+1}\downarrow H|0)\\ &\quad=N_{2k}^{(u)}(H_{2k}\uparrow H_{2k+1}|0)[1-w_d(H_{2k+1}\downarrow H)]. \end{aligned} \tag{29}$$

The expectation value of the magnetization of the system at the external magnetic field H increasing after the reversal field H_{2k} can be obtained from the equation

$$\begin{aligned} i^{-1}&\left[\frac{d\Phi_u(H_{2k}\uparrow H,\omega)}{d\omega}\right]_{\omega=0}\\ &=M_{2k}^{(u)}(H_{2k}\uparrow H|0)\\ &=2M_sN_{2k}^{(u)}(H_{2k}\uparrow H|0)-NM_s, \end{aligned} \tag{30}$$

where

$$N_{2k}^{(u)}(H_{2k}\uparrow H|0)=\left[\frac{d\Gamma_{2k}^{(u)}(H_{2k}\uparrow H,z)}{dz}\right]_{z=1}. \tag{31}$$

The recursive relation

$$\begin{aligned} N_{2k}^{(u)}(H_{2k}\uparrow H|0)&=Nw_u(H_{2k}\uparrow H)+N_{2k-1}^{(d)}(H_{2k-1}\downarrow H_{2k}|0)\\ &\quad\times[1-w_u(H_{2k}\uparrow H)] \end{aligned} \tag{32}$$

follows from the other fundamental Eq. (21).

By intoducing *the relative magnetizations*

$$m_{2k+1}^{(d)}(H_{2k+1}\downarrow H|0)=\frac{1}{NM_s}M_{2k+1}^{(d)}(H_{2k+1}\downarrow H|0) \tag{33}$$

and

$$m_{2k}^{(u)}(H_{2k}\uparrow H|0)=\frac{1}{NM_s}M_{2k}^{(u)}(H_{2k}\uparrow H|0), \tag{34}$$

from Eqs. (27), (29) and (30), (32) after elementary calculations the following recursive relations are obtained:

$$\begin{aligned} &m_{2k+1}^{(d)}(H_{2k+1}\downarrow H|0)\\ &\quad=[1+m_{2k}^{(u)}(H_{2k}\uparrow H_{2k+1}|0)][1-w_d(H_{2k+1}\downarrow H)]-1 \end{aligned} \tag{35}$$

and

$$\begin{aligned} &m_{2k}^{(u)}(H_{2k}\uparrow H|0)\\ &\quad=2w_u(H_{2k}\uparrow H)+[1+m_{2k-1}^{(d)}(H_{2k-1}\downarrow H_{2k}|0)]\\ &\qquad\times[1-w_u(H_{2k}\uparrow H)]-1. \end{aligned} \tag{36}$$

In order to solve this system of recursive equations we need the formula for *the starting branch* of the relative magnetization. Since *the negative saturation* has been chosen as the initial state of the system it follows from Eq. (9) that if $H_l\Rightarrow H_0=-\infty$, then

$$w_u(H_l\uparrow H)\Rightarrow F_u(H|\mathcal{C}),$$

and so the equation for the starting branch will be

$$m_0^{(u)}(-\infty\uparrow H|0)=2F_u(H|\mathcal{C})-1. \tag{37}$$

This branch can be also called *the limiting ascending branch* because there is no branch below it. If *the positive saturation* would be the initial state then it is easy to show that *the limiting descending branch* can be written in the form

$$m_0^{(d)}(\infty\downarrow H|0)=2F_d(H|\mathcal{C})-1, \tag{38}$$

where $F_d(H|\mathcal{C})$ is defined by the expression (8), and at the same time it is obvious that the limiting descending branch has the property that there is no other branch above it. The two limiting curves form *the major hysteresis loop* which defines an area where all other loops should be located.

By using the expression (37) for the starting branch and Eq. (35) we can obtain the first descending branch

$$m_1^{(d)}(H_1\uparrow H|0)=2F_u(H_1|\mathcal{C})[1-w_d(H_1\uparrow H)]-1, \quad (39)$$

which is attached to the limiting ascending branch at the point H_1. This descending branch is called by Mayergoyz [1] *the first-order transition curve*. The field H_1 where the first-order transition curve starts from, will be called *start field*. Denote by H_2 the next reversal point where the magnetizing field begins again to increase. The corresponding ascending branch, i.e., *the second-order transition curve* is given by the formula

$$m_2^{(u)}(H_2\uparrow H|0)=2w_u(H_2\uparrow H)+[1+m_1^{(d)}(H_1\downarrow H_2)][1-w_u(H_2\uparrow H)]-1. \quad (40)$$

This procedure can be continued and it is seen that there is no need to take into account any special requirement in order to describe the field dependence of the average magnetization since the Markov points of the magnetizing field determine exaclty the stochastic behavior of the process.

D. Stationarity of hysteresis loops

Let us investigate now the variation of the magnetization for a *special sequences of reversal fields*. Let as suppose that $H_{2k+1}=H_u$, $\forall\ k=0,1,\ldots$, while $H_{2k}=H_d$, $\forall k=1,2,\ldots$, and $H_u \geqslant H_d$, i.e., the magnetizing field is varying between two extreme values H_u and H_d. The field variation which starts with a decrease of the the external magnetic field H from the reversal point H_u until it reaches the next reversal point H_d and then turns to increase to the nearest H_u value, is called the magnetizing cycle. The magnetizing cycle results in a hysteresis loop called the minor hysteresis loop. The first cycle corresponds to the variation of the external field between the reversal points $H_1 \Rightarrow H_2 \Rightarrow H_3$, where $H_1=H_3=H_u$ and $H_2=H_d$, while kth cycle is done by the variation of the magnetizing field between the reversal points $H_{2k-1}\Rightarrow H_{2k}\Rightarrow H_{2k+1}$, where $H_{2k-1}=H_{2k+1}=H_u$ and $H_{2k}=H_d$ for $k=1,2,\ldots$. For the descending branch of the kth minor loop one can obtain from Eq. (35) the following expression:

$$m_{2k-1}^{(d)}(H_u\downarrow H|0)=[1+m_{2k-2}^{(u)}(H_d\uparrow H_u|0)][1-w_d(H_u\downarrow H)]-1. \quad (41)$$

while for the ascending branch of the kth minor loop the relation

$$m_{2k}^{(u)}(H_d\uparrow H|0)=2w_u(H_d\uparrow H)+[1+m_{2k-1}^{(d)}(H_u\downarrow H_d|0)]\times[1-w_u(H_d\uparrow H)]-1 \quad (42)$$

can be derived from Eq. (36). By using Eq. (37) we have

$$m_1^{(d)}(H_u\downarrow H|0)=2F_u(H_u|\mathcal{C})[1-w_d(H_u\downarrow H)]-1, \quad (43)$$

for the descending branch of the first minor loop and

$$m_2^{(u)}(H_d\uparrow H|0)=2F_u(H_u|\mathcal{C})[1-w_d(H_u\downarrow H_d)]\times[1-w_u(H_d\uparrow H)]+2w_u(H_d\uparrow H)-1 \quad (44)$$

for the ascending branch of the same loop. It is to note that according to the Mayergoyz's terminology the first minor loop consists of a first-order descending and a second-order ascending transition curves. Following the reversal points of the magnetizing field this procedure can be continued and we can obtain both the descending and ascending branches of relative magnetization for any minor loop.

One can prove *a very important limit theorem*, namely, there exist two *limit curves*

$$\lim_{k\to\infty} m_{2k-1}^{(d)}(H_u\downarrow H|0)=m_d(H_u\downarrow H), \quad (45)$$

and

$$\lim_{k\to\infty} m_{2k}^{(u)}(H_d\uparrow H|0)=m_u(H_u\downarrow H), \quad (46)$$

which are determining a closed minor loop. In other words, the magnetizing process becomes stationary with increasing number of cycles. It means the system ''forgets'' gradually its initial state by repeating the magnetizing cycle. This forgetting process can be related to the well-known *accommodation process*. The original Preisach model results in an *immediate formation* of the minor hysteresis loop after only one cycle of back-and-forth variation of the input between any two consecutive extremum values. However, this consequence of the Preisach model contradicts to well known experimental finding that the hysteresis loop formation is preceded by an accommodation process which can be sometimes appreciable [10,11]. In order to describe this accommodation process the traditional Preisach model was modified in the ''moving'' and the ''product'' models [12]. One has to mention that the modification of the moving model performed by Mayergoyz [1] (see Sec. II. 5, pp. 108–114) gives not only a possible variant of the accommodation but defines a sufficient condition too for the convergence of the process. The stochastic model developed by us contains the phenomenon of accommodation inherently as a consequence of the limit theorem (45) and (46).

The formulas for the limit curves defined by Eqs. (45) and (46) can be obtained by some elementary calculations as follows:

$$m_d(H_u\downarrow H)=2Q(H_d,H_u)w_u(H_d\uparrow H_u)[1-w_d(H_u\downarrow H)]-1 \quad (47)$$

and

$$m_u(H_d\uparrow H)=1-2Q(H_d,H_u)w_d(H_u\downarrow H_d)[1-w_u(H_d\uparrow H)], \quad (48)$$

where

$$Q(H_d,H_u)=[w_d(H_u\downarrow H_d)+w_u(H_d\uparrow H_u)-w_d(H_u\downarrow H_d)w_u(H_d\uparrow H_u)]^{-1}.$$

From these equations two important relations can be derived, namely,

$$m_d(H_u\downarrow H_u)=m_u(H_d\uparrow H_u) \text{ and } m_d(H_u\downarrow H_d)=m_u(H_d\uparrow H_d),$$

which show that the return-point memory property is fulfiled for the accommodated minor hysteresis loops. It is also obvious, that the accommodated minor loops due to the same pair of reversal fields H_d and $H_u \geqslant H_d$ are not only congruent but identical since the field values H_d and H_u unambiguously determine the the branches of stationary loops. One has to mention that the accommodated branches $m_d(\infty\downarrow H)$ and $m_u(-\infty\uparrow H)$ are exactly identical with the limiting descending and ascending branches which indicates the consistency of the theory.

The explicit form of the expressions $m_d(H_u\downarrow H)$ and $m_u(H_d\uparrow H)$ which describe the descending and the ascending branches of the stationary minor loop between two reversal fields H_d and $H_u \geqslant H_d$ has a great advantage in numerical calculations in comparision with the well-known Everett integral. It is to be noted that the expressions (47) and (48) are suitable to describe not only symmetrical but *asymmetrical hysteresis loops* too and it is easy to show that symmetrical hysteresis loops can be obtained only if the function $h(x,y)$ has a *mirror symmetry* expressed by

$$h(x,y)=h(-y,-x). \tag{49}$$

In the following the mirror symmetry of $h(x,y)$ will be assumed.

It is worthwhile to derive the formula for *the virgin curve of the magnetization* depending on the parameters of the density function $h(x,y|\mathcal{C})$. After some simple manipulations we obtain

$$m_0(H)=2\frac{s_1(H)}{s_1(H)+s_2(H)-s_1(H)s_2(H)}-1, \tag{50}$$

where

$$s_1(H)=\frac{\int_{-H}^{+H}dx\int_{-\infty}^{x}h(x,y)dy}{\int_{-H}^{+\infty}dx\int_{-\infty}^{x}h(x,y)dy} \tag{51}$$

and

$$s_2(H)=\frac{\int_{-H}^{+H}dy\int_{y}^{+\infty}h(x,y)dx}{\int_{-\infty}^{+H}dy\int_{y}^{+\infty}h(x,y)dx}. \tag{52}$$

If the function $h(x,y)$ satisfies the symmetry relation (49) then it is easy to prove that

$$\lim_{H\to 0} m_0(H)=0$$

and *the initial susceptibility* defined by

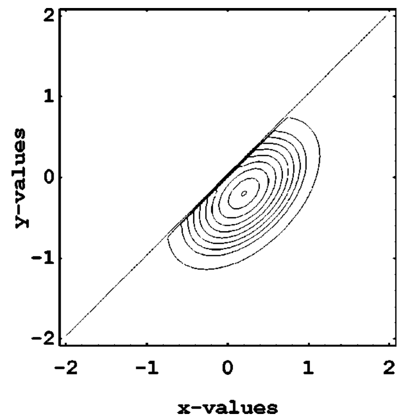

FIG. 3. Contour plot of $h(x,y|\mathcal{C})$ defined by Eq. (6) where $h(x,y)$ is given by Eq. (54) with parameter values $H_c=0.2,\sigma=0.6$, and $C_r=0.5$.

$$\chi_a=\lim_{H\to 0}\frac{dm_0(H)}{dH}=\frac{\int_0^{+\infty}h(x,0)dx}{\int_0^{+\infty}dx\int_{-\infty}^{x}h(x,y)dy} \tag{53}$$

is different from zero in contrary to the classical Preisach model which gives a nonrealistic zero slope of the virgin curve at $H=0$.

IV. NUMERICAL CALCULATIONS AND DISCUSSION

In order to compute the magnetization vs field curves we have to know the joint density function $h(x,y|\mathcal{C})$ of the U and D fields. Since these fields are the sum of many small random components it is reasonable to assume that the central limit theorem is approximately valid and so the function $h(x,y)$ in $h(x,y|\mathcal{C})$ can be chosen in the form

$$h(x,y)=\frac{1}{2\pi\sigma^2\sqrt{1-C_r^2}}\exp\left\{-\frac{1}{2\sigma^2(1-C_r^2)}[(x-H_c)^2 -2C_r(x-H_c)(y+H_c)+(y+H_c)^2]\right\}, \tag{54}$$

where the meaning of the constants H_c,σ, and C_r is clear from the elements of the probability theory. Figure 3 shows the contour plot of $h(x,y|\mathcal{C})$ defined by Eq. (6) for the parameters $H_c=0.2,\sigma=0.6$, and $C_r=0.5$. The contours are belonging to the following values of $h(x,y|\mathcal{C})=0.1$, 0.2; 0.3(0.05)0.65 and 0.682. The last one is slightly smaller than $\max_{(x,y)}h(x,y|\mathcal{C})=0.682923\cdots$. The discontinouity along the line $y-x=0$ can be clearly seen in the figure. In the sequel this formula will be used in all of our numerical calculations provided that the correlation coefficient C_r is equal to zero, i.e., $h(x,y)=f(x)f(-y)$ where f is the density function of the normal distribution. This case corresponds to *the product model* introduced by Biorci and Pescetti [13] and used consequently by Kádár [5,6,12].

The relative magnetization vs field curves are shown in Fig. 4. The parameter values used for the calculation are $H_c=0.4,\sigma=0.6$, and $C_r=0$. The values of magnetizing field are given here and in the further figures in a properly chosen

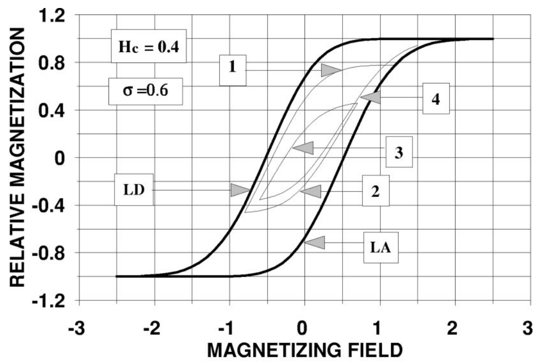

FIG. 4. Limiting branches LA, LD, and magnetization versus field curves starting from the reversal points $H_1=1.2$, $H_2=-0.8$, $H_3=0.7$, and $H_4=-0.6$. The magnetization curves are indicated by 1, 2, 3, and 4.

arbitrary unit. The curves LA and LD correspond to the limiting ascending and descending branches, while the curves indexed in the figure by 1, 2, 3, 4 are the first-, second-, third-, and fourth-order transition curves defined by the reversal points $H_1=1.2,H_2=-0.8,H_3=0.6,H_4=-0.6$. It is worth noting that the all information about the past history of the magnetizing process is transfered by the state of the system in the last reversal point. For example, the fourth-order transition curve 4 which is plotted in the field interval $[-0.6,1.5]$, is determined by the state in the reversal point $H_4=-0.6$.

It is well known that in the traditional Preisach model the minor loops which describe the cyclic change of the magnetization with back-and-forth variation of the magnetizing field between the same two limiting values are congruent and the formation of a closed minor loop is realized in one cycle, i.e., *the accommodation process* is absent. In contrast to this the stochastic model contains inherently the accommodation process which is clearly demonstrated in Fig. 5. For the sake of orientation the limiting ascending branch LA is also plotted in Fig. 5 where it is seen that the descending branch of the first minor loop starts from the point A due to the first reversal field $H_1=0.8$ and after reaching the reversal point $H_2=H_d=-0.2$ it turns to increase to the point B which corresponds to the next reversal field $H_3=H_u=0.8$. One can observe that *the first minor loop* is not closed, the point B where the decreasing branch of *the second minor loop* starts from, occupies a higher position than the point A, and the end point C of the increasing branch of the second minor loop is found above the point B but the distance between the points C and B is smaller than that between the points B and A. By repeating the magnetizing cycle between the reversal fields $H_d=-0.2$ and $H_u=0.8$ the difference between the branches of the same type becomes gradually negligible, i.e., the branches converge to *limit curves* which form finally a *closed stationary hysteresis loop* denoted by LC. The magnetizing curves measured by Carter and Richards [11] on silicon steel (4.3% Si) are surprisingly similar to that plotted in Fig. 5.

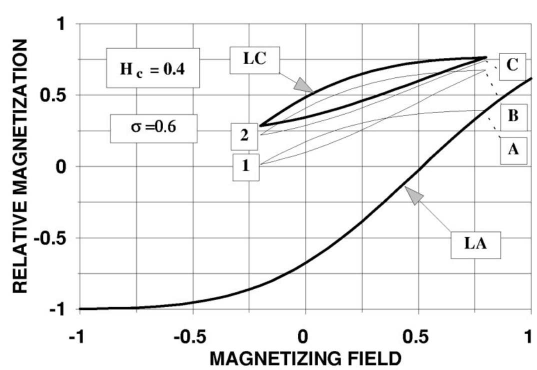

FIG. 5. Accommodation process of the minor loop in consecutive magnetizing cycles between the reversal points $H_d=-0.2$ and $H_u=0.8$. The loop LC is the accommodated minor loop.

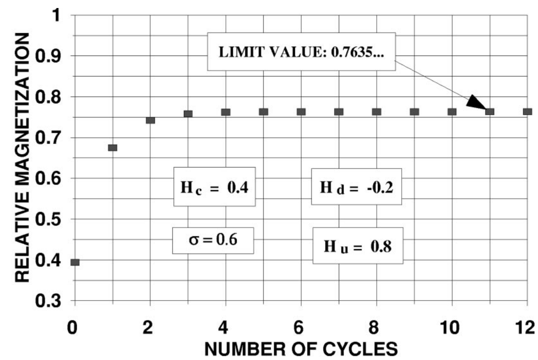

FIG. 6. Convergence of the relative magnetization in the reversal point $H_u=0.8$ with increasing number of cycles to the limit (i.e., the stationary) value.

In order to demonstrate the speed of the convergence, the nonaccommodated relative magnetizations have been calculated in the reversal point $H_u=0.8$ for the subsequent cycles. Figure 6 shows that the stationary (i.e., the limit) value of the magnetization can be very well approached by repeating the cycle 8–9 times in the case of parameter values $H_c=0.4,\sigma=0.6$, and $H_u=0.3$.

The nonaccommodated minor loops due to the same pair of reversal fields are evidently not congruent and generally are not closed. However, this noncongruency has nothing in common with that introduced and discussed in details by Kádár [5,15]. The noncongruency of the nonaccommodated minor loops bounded by the same field limits has a quite different origin in the stochastic model, namely, the nonequilibrium response of the system for the cyclic back-and-forth variation of the external magnetic field between two consecutive reversal points. It is obvious consequence of the non-stationarity of minor loops that the return-point memory property is absent in these loops.

In order to study the properties of noncongruency of this type the first minor loops belonging to different start fields are calculated. Denote by $\Delta H=H_u-H_d$ the difference between the consecutive reversal fields. For the characterization of the nonaccommodated first minor loops due to different start fields H_1 let us introduce two parameters defined by

$$\mathrm{W}=W(H_1,\Delta H)=\max_{H_1-\Delta H\leqslant H\leqslant H_1}[m_1(H_1\downarrow H)$$

$$-m_2(H_1-\Delta H\uparrow H)]$$

and

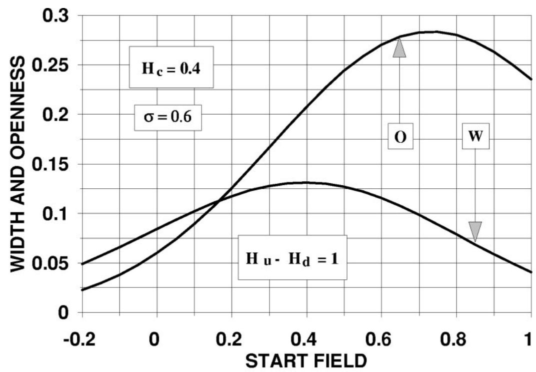

FIG. 7. Width W of the first-order minor loops and the difference O between the values of the descending and ascending branches in the reversal point $H_u = 0.8$ versus start field.

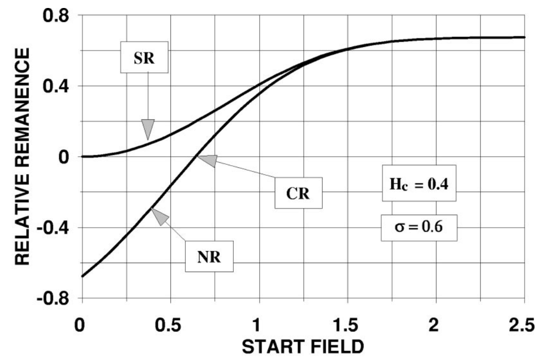

FIG. 9. The nonaccommodated NR and the stationary SR relative remanences versus start field due to different points of the ascending limiting branch.

$$O = O(H_1, \Delta H) = m_2(H_1 - \Delta H \uparrow H_1) - m_1(H_1 \downarrow H_1).$$

The dependence of these parameters on the start field H_1 is shown in Fig. 7 for the parameter values $H_d = -0.2, \Delta H = 1$. The author of the present paper is far not convinced whether the experimental data contradict or support the noncongruency of this type because of the lack of careful measurements.

It seems to be useful to investigate the remanence properties of systems described by the stochastic model. In Fig. 8 the first-order descending curves which start from different points of the ascending limiting branch LA can be seen. The curves starting from the points due to the field values $H_1 = 1.6, H_2 = 1.4, H_3 = 1.2, H_4 = 1$ are plotted to the points of remanences R1, R2, R3, R4 which are obviously different from the stationary (i.e., the accommodated) values. The nonaccommodated NR and stationary remanences SR versus start field are shown in Fig. 9. As it is seen the nonaccommodated remanences can be negative below a critical start field CR since the initial negative saturation has a significant effect on the first-order transition curves. The stationary remanence curve SR calculated from Eqs. (47) and (48) is non-negative in all points of the start field interval.

The *influence of the parameter* σ on the shape of the major hysteresis loop can be seen in Fig. 10. As it is expected the larger is the parameter σ the wider is the hysteresis loop, i.e., the larger nonhomogeneity in a system (e.g., in a magnetic sample) results in a higher ‘‘coercive force.’’

It seems to be useful to calculate the *accommodated (stationary) hysteresis loops* for different pairs of reversal points H_d and $H_u \geq H_d$. The hysteresis loops plotted in Fig. 11 correspond to the reversal points $H_d = -1.5, H_u = 1.5$ (loop ML1), $H_d = -1, H_u = 1$ (loop ML2), $H_d = -0.5, H_u = 0.5$ (loop ML3). For the calculation we used the parameter values $H_c = 0.4$ and $\sigma = 0.6$. For the sake of completeness *the virgin curve* VC calculated by Eq. (50) and the major loop LL bounded by the limiting ascending and descending curves are also shown in the figure. The stochastic model clearly shows that all accommodated minor loops corresponding to cyclic inputs between the same two consecutive extremum values are not only congruent but simply identical.

In Fig. 12 three accommodated first-order minor loops denoted by 1, 2, 3 can be seen. The descending branches of the loops are started from the field values $H = 0.5$, 0.3, 0, and each of the ascending branches returns exactly to the same point that the corresponding descending branch left. The returning curves have an apparent slope discontinuity with regard to the major loop ALA.

At this point it is worth to make a remark of somewhat historical nature. As it is well-known Preisach's idea for his model was originated from the *quadratic Rayleigh relation* which can be easily obtained [14] assuming a uniform distribution of the U and D fields over the ‘‘Preisach triangle.’’ It is interesting to note that in the stochastic model the calculated hysteresis loops almost perfectly coincide with that calculated by the Rayleigh formula when the reversal fields H_d and $H_u \geq H_d$ and so the magnetizing field $H \in [H_d, H_u]$

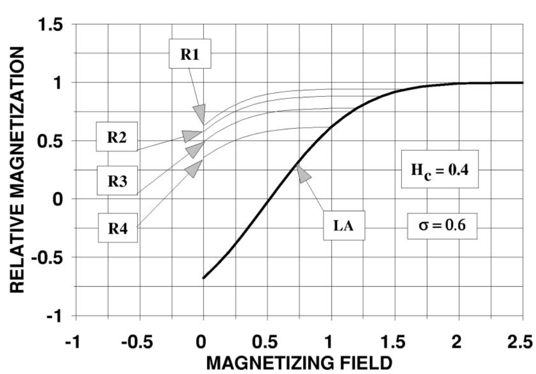

FIG. 8. The limiting ascending branch **LA** and four first-order descending curves ending in nonaccommodated remanences denoted by **R1, R2, R3**, and **R4**.

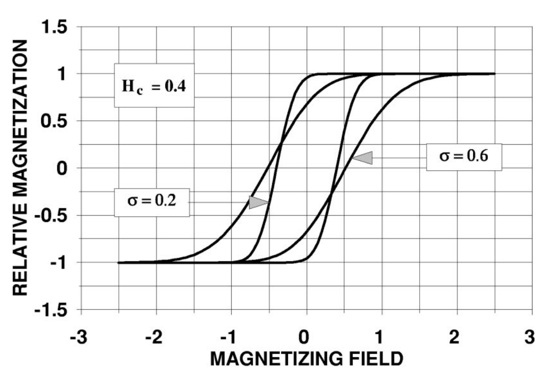

FIG. 10. Influence of the parameter σ on the shape of the major hysteresis loop in the case of $H_c = 0.4$.

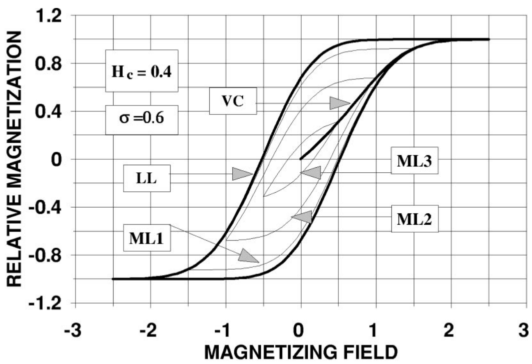

FIG. 11. The virgin curve VC, the major loop LL and three accommodated hysteresis loops ML1, ML2, ML3 calculated for different pairs of reversal points in the case of parameter values $H_c=0.4$ and $\sigma=0.6$.

are sufficiently small. The hysteresis loop R defined by reversal points $H_u=0.5$ and $H_d=-0.5$ in Fig. 13 can be very well approximated by the equations

$$m_a^d(0.5\downarrow H)=C_0^{(d)}+C_1^{(d)}H+C_2^{(d)}H^2,$$

$$m_a^u(0.5\uparrow H)=C_0^{(u)}+C_1^{(u)}H+C_2^{(u)}H^2,$$

where

$$C_0^{(d)}=-C_0^{(u)}=0.13035\cdots,$$

$$C_1^{(d)}=C_1^{(u)}=0.79301\cdots,$$

$$C_2^{(d)}=-C_2^{(u)}=-0.54251\cdots$$

in the case of parameter values $H_c=0.2$ and $\sigma=0.6$. In Fig. 13 the squares correspond to the values calculated by the quadratic equations. The excellent aggreement with the curves of the stochastic model indicates that the Rayleigh law can be reproduced in a straightforward way in the stochastic model.

This model differs from the original Preisach model in a very essential point in relation to the *reversal point susceptibility*. Namely, the nonzero initial susceptibility at the turning points is an inherent property of the stochastic model, while the traditional Preisach model can produce positive initial slope only if the Preisach function is supposed to have

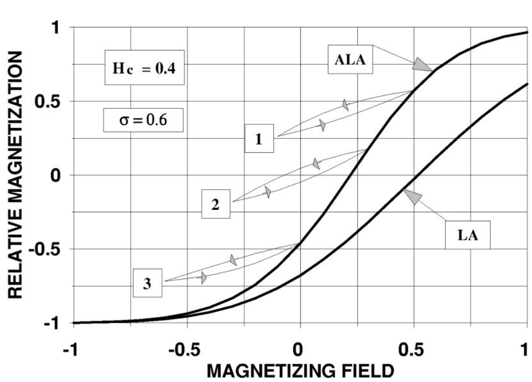

FIG. 12. Three accommodated first-order minor loops denoted by 1, 2, 3 and the shifted ascending branch ALA.

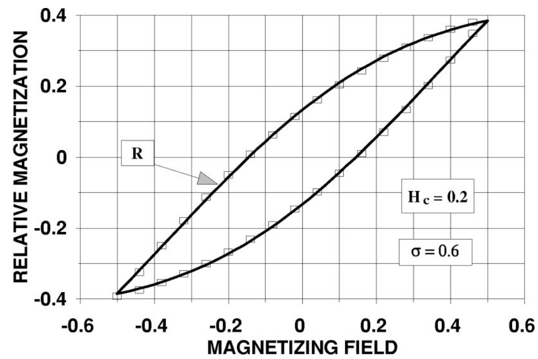

FIG. 13. The hysteresis loop between ''small'' reversal points and the quadratic Rayleigh curves denoted by squares.

a Dirac-delta-like singularity along the boundery of the Preisach triangle, as it was shown by Mayergoyz [1]. The susceptibility vs magnetizing field is seen in Fig. 14 for the parameter values $H_c=0.2$ and $\sigma=0.6$. The shape of the calculated curve can be expected on the basis of physical considerations and corresponds to those found experimentally.

The estimation of the joint density function $h(x,y|\mathcal{C})$ from measured hysteresis curves was beyond the scope of our present theoretical consideration. Of course, one may attempt in simple cases to estimate the parameters of a plausible density function [e.g., Eq. (54)] by an appropriate data evaluation procedure.

V. CONCLUSIONS

It has been shown that the Preisach model can be improved by describing the hysteresis as a *stochastic process* defined on a set of all possible values of the control parameter the reversal (turning) points of which are Markov points of the process. The one dimensional distribution function of the stochastic process has been exactly determined and the magnetizations versus up and down magnetic fields have been calculated as expectation values of the stochastic process. It has been proven that the magnetizing process becomes stationary with increasing number of magnetizing cycles. It means that for the description of the accommodation process there is no need of any artificial auxiliary assumption since the stochastic model contains the phenomenon of accommodation inherently. In general case the model is able to describe the symmetric as well as the asymmetric hysteresis. In relatively small magnetizing fields the

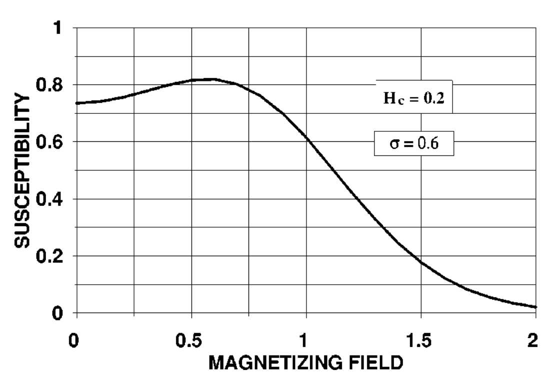

FIG. 14. The irreversible susceptibility versus magnetizing field H.

quadratic Rayleigh law can be easily obtained from the equations of the stochastic model. It is important to note that the turning point susceptibilities have nonzero finite values in contrary to the traditional Preisach model which does not take consequently into account the random nature of the elementary switching process. Finally, the stochastic model shows that all stationary loops corresponding to the same two limiting values of the magnetizing field are equivalent but the nonstationary loops are noncongruent and in general not closed.

ACKNOWLEDGMENT

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