Exact solution of the biquadratic spin-1 t-J model in one dimension

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A generalization of the t-J model with a nearest-neighbor hopping is formulated and solved exactly by the Bethe ansatz method. The model describes the dynamics of spin-S fermions with isotropic or anisotropic interactions. In the case $S=1$ the magnetic interaction is biquadratic in the spin operators. In contrast to the SU(N) generalization of the t-J model, studied previously in the literature, the present model possesses beyond a massless excitation also a massive one. The physical properties indicate the existence of Cooper-type pairs with finite binding energy. [S0163-1829(97)05538-0]

The t-J model has emerged as a paradigm for studying the low-energy electronic properties of the copper-oxide-based high-temperature superconductors.[1,2] Although high-T_c cuprates are at least two-dimensional systems the one-dimensional version of the model and its generalizations are also intensively studied since in this case exact results can be derived.[3–12] The t-J model describes the dynamics of spin-$\frac{1}{2}$ fermionic particles with Hamiltonian given by

$$H=-t\sum_{j,\sigma} P(c^+_{j,\sigma}c_{j+1,\sigma}+c^+_{j+1,\sigma}c_{j,\sigma})P + J\sum_{j} (\vec{S}_j\cdot\vec{S}_{j+1}-n_jn_{j+1}/4), \tag{1}$$

where $c_{j,\sigma}$ is the standard fermion creation operator, $\vec{S}_j=\frac{1}{2}\vec{\sigma}_j$ is the particle-spin operator and n_j is the particle-number operator at site j. The projection operator P excludes the double occupation at each site. Unfortunately the exact integrability of Eq. (1) is obtained only at the supersymmetric point $J=2t$.[3–6] At this point the model has no gap and the critical exponents governing the long-distance behavior of correlation functions were calculated.[7] These results show that for any density of holes the spin-spin correlation functions dominate the superconducting ones, and as a consequence the model has no superconducting properties. These results were extended to the SU(N) generalization of the t-J model of fermions of arbitrary spin S.[8–10] The integrability of an anisotropic generalization of the SU(N) supersymmetric t-J model has been shown[13,14] and the critical exponents of the correlation functions have been calculated.[15,16]

In this paper we present a set of models of strong-correlated electrons which are exactly solvable. The first example of these models is the spin-1 biquadratic t-J model with Hamiltonian given by

$$H=-t\sum_{j,\sigma} P(c^+_{j,\sigma}c_{j+1,\sigma}+c^+_{j+1,\sigma}c_{j,\sigma})P - J\sum_{j} [(\vec{S}_j\cdot\vec{S}_{j+1})^2-n_jn_{j+1}], \tag{2}$$

where now $\vec{S}_j=(S^x_j,S^y_j,S^z_j)$ are spin-1 Pauli operators located at site j. We show that this model is exactly integrable at the special point $t=J$. Actually the above Hamiltonian is the isotropic version of a family of anisotropic models describing the dynamics of spin-S fermions with Hamiltonian

$$H=-\sum_{j=1}^{L}\sum_{s=-S}^{S} P(c^+_{j,s}c_{j+1,s}+c^+_{j+1,s}c_{j,s})P -\varepsilon\sum_{j=1}^{L}\left[\sum_{s,t=-S}^{S} u_su_tc^+_{j,s}c_{j,t}c^+_{j+1,-s}c_{j+1,-t} -(1+\varepsilon_1)\cosh\gamma n_jn_{j+1}\right], \tag{3}$$

where L is the lattice size, $\varepsilon,\varepsilon_1=\pm1$ and the parameters u_s, which play the role of anisotropies should satisfy $u_s=1/u_{-s}$ $(s=-S,-S+1,\dots,S)$ and $2\cosh\gamma=u^2_{-S}+u^2_{-S+1}+\cdots+u^2_S$. The particular case $S=\frac{1}{2}$ and $\varepsilon=-\varepsilon_1=1$ is the anisotropic version of the supersymmetric t-J model. The biquadratic t-J model, at $t=J$, given in Eq. (2) is obtained by choosing in Eq. (3) $S=1$, $\varepsilon=-\varepsilon_1=1$ and $u_{-1}=u_0=u_1=1$. For general spin S the magnetic interactions can be written as a polynomial of degree $2S$ in the spin operators.

The exact integrability of these models, from a mathematical point of view, comes from the fact that the Hamiltonian density in Eq. (3) is related to the generators of Hecke algebras,[17] with deformation parameter q given by the relation $q+1/q=2\cosh\gamma$.

The eigenstates and eigenvalues of Hamiltonian (3) can be obtained exactly within the framework of the Bethe-ansatz method.[18–21] The structure of the Bethe-ansatz equations follows from the solution of the two-particle problem. The two-electron wave function can be written as a product of two factors: a coordinate wave function (referring to the positions and momenta of the particles) and a spin part, the global wave function being antisymmetric under the exchange of two particles. The scattering matrix can be written in the following form:

$$S^{\alpha\beta}_{\alpha',\beta'}(\lambda_1-\lambda_2)=[1+(1+\varepsilon_1)\cosh\gamma\Phi(\lambda_1-\lambda_2)]\times\delta_{\alpha,\beta'}\delta_{\beta,\alpha'}-\varepsilon_1 u_\alpha u_{\beta'}\Phi(\lambda_1-\lambda_2)\times\delta_{\alpha,-\beta}\delta_{\beta',-\alpha'}, \tag{4}$$

where

$$\Phi(\lambda)=-\frac{\sin\lambda}{\sin(\lambda-i\gamma)} \tag{5}$$

and λ_j $(j=1,2,\ldots,n)$ are suitable particle rapidities related to the momenta $\{k_j\}$ of the electrons by

$$k_j=\begin{cases}\pi-\Theta(\lambda_j;\frac{1}{2}\gamma), & \varepsilon\varepsilon_1=-1,\\ -\Theta(\lambda_j;\frac{1}{2}\gamma), & \varepsilon\varepsilon_1=+1,\end{cases} \tag{6}$$

with the function Θ defined by

$$\Theta(\lambda;\gamma)=2\arctan(\cot\gamma\tan\lambda);\quad -\pi<\Theta(\lambda,\gamma)\leqslant\pi. \tag{7}$$

A necessary and sufficient condition for the applicability of the Bethe ansatz method is the Yang-Baxter equation.[18,21] In our case the S-matrix satisfies these equations in the nondeformed and q-deformed cases.[17] The isotropic case corresponds for $S>\frac{1}{2}$ to the q-deformed case where $u_s=1(s=-S,\ldots,S)$ and $q+1/q=2S+1$. The underlying Hecke algebra of the model implies that differently from the supersymmetric t-J model we should have gapped spin excitations for $S\geqslant 1$. This model is an example of an integrable model with the S-matrix of the form (4) which is connected with the Hecke algebra. The Hamiltonian (3) is diagonalized by standard procedure by imposing periodic boundary conditions on the Bethe function. These boundary conditions can be expressed in terms of the transfer matrix of the nonuniform model which can be constructed on the basis of the S-matrix (4) by using the quantum method of the inverse problem.[22,23] The rapidities $\{\lambda_j\}$ that define a n-particle wave function are obtained by solving the equations

$$\left[\frac{\sinh(\lambda_j-i\gamma/2)}{\sinh(\lambda_j+i\gamma/2)}\right]^L=(-1)^{n-1}\Lambda(\lambda_j), \tag{8}$$

where $\Lambda(\lambda)$ is the eigenvalue of the transfer matrix

$$T^{\{\alpha_l\}}_{\{\alpha'_l\}}(\lambda)=\sum_{\{\beta_l\}}\prod_{l=1}^{n}S^{\alpha_l\beta_{l+1}}_{\alpha'_l\beta_l}(\lambda_l-\lambda),\ (\beta_{n+1}=\beta_1). \tag{9}$$

It is simple to verify that besides the number of particles n, the magnetization $\Sigma_j S^z_j$ and the number of paired electrons m are conserved quantities in the Hamiltonian (3). Two electrons are paired if they are consecutive electrons with opposite spins and have no unpaired electron between them. The complete diagonalization of the transfer matrix (9) is not a simple problem even in the simplest case $S=1,n=L$ (see, for example, Ref. 24). It is not difficult to convince ourselves that in the interesting physical situation where we have low density of holes the ground state will belong to the sector where we have zero magnetization and only pairs of electrons. In this sector $m=n/2$ and the diagonalization of the transfer matrix of the inhomogeneous model (9) gives for $\varepsilon_1=-1$, the following equations:

$$\left[\frac{\sin(\lambda_j+i\gamma/2)}{\sin(\lambda_j-i\gamma/2)}\right]^L=(-1)^{m-1}\prod_{\alpha=1}^{m}\frac{\sin(\lambda_j-\Lambda_\alpha+i\gamma/2)}{\sin(\lambda_j-\Lambda_\alpha-i\gamma/2)}, \tag{10}$$

$$\prod_{j=1}^{n}\frac{\sin(\lambda_j-\Lambda_\alpha-i\gamma/2)}{\sin(\lambda_j-\Lambda_\alpha+i\gamma/2)}=-\prod_{\beta=1}^{m}\frac{\sin(\Lambda_\alpha-\Lambda_\beta+i\gamma)}{\sin(\Lambda_\alpha-\Lambda_\beta-i\gamma)}.$$

In the case $\varepsilon_1=+1$ the first set of equations in Eq. (10) should be replaced by

$$\left[\frac{\sin(\lambda_j+i\gamma/2)}{\sin(\lambda_j-i\gamma/2)}\right]^L=(-1)^{m-1}\prod_{l=1}^{n}\frac{\sin(\lambda_j-\lambda_l+i\gamma)}{\sin(\lambda_j-\lambda_l-i\gamma)}\times\prod_{\alpha=1}^{m}\frac{\sin(\lambda_j-\Lambda_\alpha-i\gamma/2)}{\sin(\lambda_j-\Lambda_\alpha+i\gamma/2)}.$$

The total energy and momentum of the model are given in terms of the particle rapidities λ_j in the following form:

$$E=-2\sum_{j=1}^{n}\cos k_j=2\varepsilon\varepsilon_1\sum_{j=1}^{n}\left(\cosh\gamma-\frac{\sinh^2\gamma}{\cosh\gamma-\cos 2\lambda_j}\right), \tag{11}$$

$$P=\sum_{j=1}^{n}k(\lambda_j).$$

Equations (10) and (11) have the same structure as those appearing in the anisotropic t-J model[15,16] provided a suitable definition of the parameter γ is given. It means that in spite of the physical processes in the models with $S=\frac{1}{2}$ and $S>\frac{1}{2}$ being quite different there is a "weak equivalence" in Baxter's sense[25] between models with different values of spin S in the sector where $m=n/2$. Of course in the general case this equivalence does not exist.

Although the models are exactly integrable for both signs of ε and ε_1 in Eq. (3) let us now restrict to the more physically interesting case $\varepsilon=1$ and $\varepsilon_1=-1$, where we have attraction among pairs. In this case the ground state contains $m=n/2$ bound pairs characterized by a pair of complex electron rapidities

$$\lambda^{\pm}_\alpha=\frac{1}{2}(v_\alpha\pm i\gamma),\quad v_\alpha=2\Lambda_\alpha. \tag{12}$$

The second set of equations in Eq. (10) is fullfilled within exponential accuracy whereas the first set can be treated in the similar way as in Refs. 15, 16. Inserting Eq. (12) in the first set of equations in Eq. (10) and introducing the density function $\rho(v)$ for the distribution of v_α in the thermodynamic limit, we obtain the linear integral equation

$$2\pi\rho(v)=\Phi(v;\gamma)-\int_I\Phi(v-v';\gamma)\rho(v')dv', \tag{13}$$

where

$$\Phi(v;\gamma)=\frac{\sinh 2\gamma}{\cosh 2\gamma-\cosh v}. \tag{14}$$

In order to minimize the ground-state energy

$$\frac{E_0}{L}=-2\varepsilon\int_I[2\cosh\gamma-\sinh\gamma\Phi(v;\gamma)]\rho(v)dv, \tag{15}$$

the integration interval I in Eqs. (13) and (15) has to be chosen symmetrically around $\pi (I=[v_0, 2\pi - v_0])$. The parameter v_0 is determined by the subsidiary condition for the total density $\rho = 2m/L$ of electrons

$$\int_I \rho(v) dv = \frac{1}{2}\rho. \tag{16}$$

To study the superconducting properties of the model under consideration we calculate the long-distance behavior of the correlation functions by finite-size studies and application of conformal field theory (see Refs. 26–28, and references therein). The results of this calculation are the following. The long-distance behavior of the density-density and the superconducting correlation functions are given by

$$\langle \rho(r)\rho(0)\rangle \simeq \rho^2 + A_1 r^2 + A_2 r^{-\alpha}\cos(2k_F r); \quad 2k_F = \pi\rho; \tag{17}$$

$$\rho(r) = \sum_\gamma c^+_{r\gamma} c_{r\gamma},$$
$$G_\rho(r) = \langle c^+_{r\gamma} c^+_{r+1,-\gamma} c_{0,\delta} c_{1,-\delta}\rangle \simeq B r^{-\beta}. \tag{18}$$

The exponents α and β describing the algebraic decay are calculated from the dressed charge function $\xi(v)$ which is given by the solution of the integral equation

$$\xi(v) = 1 - \frac{1}{2\pi}\int_I \Phi(v - v'; \gamma)\xi(v') dv', \tag{19}$$

and is given by

$$\alpha = \beta^{-1} = 2[\xi(v_0)]^2. \tag{20}$$

In our one-dimensional system we have no superconductivity in the literal sense, since the model does not have finite off-diagonal long-range order. But we may say that in our model there is a tendency to superconductivity since the superconducting correlations have a longer range than the density-density correlations. This happens when $\beta < \alpha$. Analytically we find $\alpha = 2$ for $(\rho = 0)$ and $\alpha = \frac{1}{2}$ for $(\rho = \rho_{\max} = 1)$. This implies that for all nonzero values of the parameters γ there is a density regime $[0, \rho_c]$ where the system has dominating superconducting correlations. An analogous behavior of correlation functions can also be observed in the SU(N) generalization of the anisotropic t-J model where superconducting properties are caused by the introduction of anisotropy in the interactions. However unlike these models the superconducting properties in the Hamiltonians (3) are caused by both effects, the anisotropy and the value of the spin S [see definition of the parameter γ Eq. (3)]. Moreover in the present model for any value of N ($N = 2S+1$) we have bound pairs but not complexes of N bound particles as in Ref. 16.

We conclude this paper with some remarks about the lattice vertex model counterpart of the quantum chain considered here. The quantum R matrix has $1 + 3N + 2N^2$ nonzero Boltzmann weights, which are given by

$$R^{00}_{00} = 1, \;\; R^{0\alpha}_{0\alpha} = R^{\alpha 0}_{\alpha 0} = \varepsilon \;\sinh\lambda / \sinh(\gamma - \varepsilon_1 \lambda),$$
$$R^{0\alpha}_{\alpha 0} = R^{\alpha 0}_{0\alpha} = \sinh\gamma / \sinh(\gamma - \varepsilon_1\lambda), \tag{21}$$
$$R^{\alpha\beta}_{\gamma\delta} = [\delta_{\alpha,\delta}\delta_{\beta,\gamma} + \Phi(i\lambda) u_\alpha u_\delta \delta_{\alpha, N-\beta+1}\delta_{\delta, N-\gamma+1}] \times \sinh(\gamma - \lambda)/\sinh(\gamma - \varepsilon_1\lambda),$$

where $\alpha, \beta = 1, 2, \ldots, N$. The associated spin Hamiltonian can be found by taking the logarithmic derivative of the row-to-row transfer matrix at $\lambda = 0$. It gives the Hamiltonian (3) after a Jordan-Wigner transformation. Since we verified that Eqs. (21) satisfy the Yang-Baxter equations, the exact integrability of Eq. (3) is an immediate consequence. The above vertex model can be treated by the diagonal-to-diagonal Bethe ansatz method,[29,30] But this is not the aim of this work.

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